

Basics of Integration

chapter 7

Formulas
Properties
Examples

(अवकलन)

Reverse process
of Differentiation

Differentiation

$$\frac{d(x^2)}{dx} = 2x$$

$$\frac{d(x^2 + 1)}{dx} = 2x$$

$$\frac{d(x^2 + c)}{dx} = 2x \quad (c \in \mathbb{R})$$

Integration

$$\int 2x \cdot dx = x^2 + c$$

Arbitrary Constant

Integrat. Constant

$$\int f(x) \cdot dx = F(x) + c$$

Integrand

Variable
of
Int.

Antiderivative
(Primitive)

integration of $f(x)$ with
respect to 'x'

Indefinite
Integration

Geometrical meaning of Integration (Indefinite Integral)

$$\frac{d(f(x))}{dx} = f'(x)$$

Function ↑ Slope

$$\boxed{f(x) + c} \xrightarrow{\text{Diff.}} \boxed{f'(x)}$$

Functions Slope

$$\int f'(x) \cdot dx = f(x) + c$$

Slope

Functions

अवकलन

Family of
Curves

Derivatives

$$(1) \frac{d}{dx} \left(\frac{x^{n+1}}{n+1} \right) = x^n$$

$$(2) \frac{d}{dx} (x) = 1$$

$$(3) \frac{d}{dx} (\log|x|) = \frac{1}{x}$$

$$(4) \frac{d}{dx} (e^x) = e^x$$

$$(5) \frac{d}{dx} \left(\frac{a^x}{\log a} \right) = a^x$$

$$(6) \frac{d}{dx} (\sin x) = \cos x$$

$$(7) \frac{d}{dx} (-\cos x) = \sin x$$

$$(8) \frac{d}{dx} (\tan x) = \sec^2 x$$

$$(9) \frac{d}{dx} (\sec x) = \sec x \cdot \tan x$$

$$(10) \frac{d}{dx} (-\cot x) = \operatorname{cosec}^2 x$$

$$(11) \frac{d}{dx} (-\operatorname{cosec} x) = \operatorname{cosec} x \cdot \cot x$$

Anti-Derivatives (Primitives) (Integrals)

$$\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + C, \quad \underline{n \neq -1}$$

$$\int dx = \underline{x + C}$$

$$\int x^{-1} \cdot dx = \int \frac{1}{x} \cdot dx = \log|x| + C$$

$$\int e^x \cdot dx = e^x + C$$

$$\int a^x \cdot dx = \frac{a^x}{\log a} + C$$

$$\int \underline{\cos x} \cdot dx = \sin x + C$$

$$\int \underline{\sin x} \cdot dx = -\cos x + C$$

$$\int \underline{\sec^2 x} \cdot dx = \underline{\tan x} + C$$

$$\int \underline{\sec x \cdot \tan x} \cdot dx = \underline{\sec x} + C$$

$$\int \operatorname{cosec}^2 x \cdot dx = -\cot x + C$$

$$\int \underline{\operatorname{cosec} x \cdot \cot x} \cdot dx = \underline{-\operatorname{cosec} x} + C$$

$$\begin{aligned} \textcircled{12} \quad \frac{d}{dx}(\sin^{-1}x) &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}(-\cos^{-1}x) &= \frac{1}{\sqrt{1-x^2}} \end{aligned} \quad \left. \vphantom{\frac{d}{dx}} \right\} \rightarrow \begin{aligned} \int \frac{1}{\sqrt{1-x^2}} \cdot dx &= \sin^{-1}x + C \\ &= -\cos^{-1}x + C \end{aligned}$$

$$\begin{aligned} \textcircled{13} \quad \frac{d}{dx}(\tan^{-1}x) &= \frac{1}{1+x^2} \\ \frac{d}{dx}(-\cot^{-1}x) &= \frac{1}{1+x^2} \end{aligned} \quad \left. \vphantom{\frac{d}{dx}} \right\} \rightarrow \begin{aligned} \int \frac{1}{1+x^2} \cdot dx &= \tan^{-1}x + C \\ &= -\cot^{-1}x + C \end{aligned}$$

$$\begin{aligned} \textcircled{14} \quad \frac{d}{dx}(\sec^{-1}x) &= \frac{1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx}(-\operatorname{cosec}^{-1}x) &= \frac{1}{|x|\sqrt{x^2-1}} \end{aligned} \quad \left. \vphantom{\frac{d}{dx}} \right\} \rightarrow \begin{aligned} \int \frac{dx}{|x|\sqrt{x^2-1}} &= \sec^{-1}x + C \\ &= -\operatorname{cosec}^{-1}x + C \end{aligned}$$

Also Remember

$$\int x \cdot dx = \frac{x^2}{2} + C$$

$$\int \sqrt{x} \cdot dx = \frac{2}{3} x^{3/2} + C$$

$$\int \frac{1}{\sqrt{x}} \cdot dx = 2\sqrt{x} + C$$

Properties of Integration.

Diff Int

$$\textcircled{\text{I}} \quad \int \underline{f'(x)} \cdot dx = \underline{f(x)} + C$$

$$\frac{d}{dx} \left(\int f(x) \cdot dx \right) = f(x)$$

$$\begin{aligned} \textcircled{\text{II}} \quad \int \{ \underline{f(x)} \pm \underline{g(x)} \} \cdot dx &= \underline{\int f(x) \cdot dx} \pm \underline{\int g(x) \cdot dx} \\ &= \underline{F(x) + C_1} \pm (\underline{G(x)} + C_2) \\ &= F(x) \pm G(x) + \underline{C_1 \pm C_2} \rightarrow C \\ &\quad \text{New Constant.} \end{aligned}$$

$$\textcircled{\text{III}} \quad \int K \cdot \underline{f(x)} \cdot dx = \underline{K} \int \underline{f(x)} \cdot dx \quad (\underline{K = \text{constant}})$$

~~Not~~

Not all functions are integrable

$$\underline{\int e^{x^2} \cdot dx} \quad \times \quad \underline{\int e^{-x^2} \cdot dx}$$

$$\underline{\int f(x) \cdot g(x)} \quad \times$$

$$\underline{\int u \cdot v}$$

अभी नहीं

e.g. write an antiderivative of $\cos 2x$
by inspection.

Ans.

$$\int \cos 2x \cdot dx = \boxed{} + C$$

Diff.

$$\frac{d(\sin 2x)}{dx} = \cos 2x \cdot (2) = \underline{2 \cos 2x}$$

$$\Rightarrow \frac{1}{2} \frac{d(\sin 2x)}{dx} = \cos 2x$$

$$\Rightarrow \frac{d\left(\frac{1}{2} \sin 2x\right)}{dx} = \cos 2x$$

Anti derivative of $\cos 2x = \frac{1}{2} \sin 2x$

e.g. Integrate.

$$\textcircled{1} \int \left[\left(\frac{x^4 - x^2}{x^3 - x} \right) \cdot \sqrt{x} + e^x \right] \cdot dx$$

$$= \int \left(\frac{x^x(x^2-1)}{x(x^2-1)} \cdot \sqrt{x} + e^x \right) \cdot dx$$

$$= \int (\underline{x^1} \cdot \sqrt{x} + e^x) \cdot dx$$

$$= \int (x^{3/2} + e^x) \cdot dx = \frac{x^{3/2+1}}{3/2+1} + e^x + C = \frac{2x^{5/2}}{5} + e^x + C$$

$$\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + C$$

(I)

$$\int \underline{\csc x} (\underline{\csc x + \cot x}) \cdot dx$$

$$= \int (\underline{\csc^2 x} + \underline{\csc x \cdot \cot x}) \cdot dx$$

$$= -\cot x - \csc x + C$$

(II)

$$\int \frac{1 - \sin x}{\cos^2 x} \cdot dx$$

$$= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) \cdot dx$$

$$= \int (\underline{\sec^2 x} - \underline{\tan x \cdot \sec x}) \cdot dx$$

$$= \tan x - \sec x + C$$

Exercise 7.1

→ Integration using
Simple formulas

Find an anti derivative (or Integral) of the following functions by method of inspection.

[Q.1] $\sin 2x$

$$\frac{d(\cos 2x)}{dx} = -\sin 2x \cdot 2$$

$$\Rightarrow -\frac{1}{2} \frac{d(\cos 2x)}{dx} = \sin 2x$$

$$\Rightarrow \frac{d(-\frac{1}{2} \cos 2x)}{dx} = \sin 2x$$

जो given function है
वो किसके Differentiation
से आया होगा?

derivative of $-\frac{1}{2} \cos 2x$
= $\sin 2x$

Antiderivative of $\sin 2x$
= $-\frac{1}{2} \cos 2x$

[Q.2] Antiderivative of $\cos 3x = \frac{1}{3} \sin 3x$

$$\frac{d(\sin 3x)}{dx} = \cos 3x \cdot 3$$

$$\Rightarrow \frac{d(\frac{1}{3} \sin 3x)}{dx} = \cos 3x$$

[Q.3] Anti derivative of $e^{2x} = \frac{e^{2x}}{2}$

$$\frac{d(e^{2x})}{dx} = 2 \cdot e^{2x}$$

$$\Rightarrow \frac{d(\frac{e^{2x}}{2})}{dx} = e^{2x}$$

Q.4

$$(ax+b)^2$$

Anti derivative of $(ax+b)^2$

$$\frac{d}{dx} (ax+b)^3 = 3(ax+b)^2 \cdot a$$

$$= \frac{(ax+b)^3}{3a}$$

$\Rightarrow \frac{d}{dx} \left(\frac{(ax+b)^3}{3a} \right) = (ax+b)^2$

Derivative

Anti derivative

Q.5

Anti derivative of $(\sin 2x - 4e^{3x}) = -\frac{1}{2} \cos 2x - 4 \left(\frac{e^{3x}}{3} \right)$

Antiderivative of $\sin 2x = -\frac{1}{2} \cos 2x$

Antiderivative of $e^{3x} = \frac{e^{3x}}{3}$

Integral

$= -\frac{1}{2} \cos 2x - \frac{4}{3} e^{3x}$

Find the integrals in Questions 6 to 20:

Q.6

$$\int (4e^{3x} + 1) \cdot dx$$

$$= 4 \int e^{3x} \cdot dx + \int 1 \cdot dx$$

$$= 4 \left(\frac{e^{3x}}{3} \right) + (x) + C$$

$$= \frac{4}{3} e^{3x} + x + C$$

$$\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + C$$

$$\int x^0 \cdot dx = \frac{x^{0+1}}{0+1} + C$$
$$= \frac{x^1}{1} + C$$

$$\begin{aligned} \text{Q.7} \quad & \int x^2 \left(1 - \frac{1}{x^2}\right) \cdot dx \\ &= \int (x^2 - 1) \cdot dx \\ &= \frac{x^3}{3} - x + C \end{aligned}$$

$$\begin{aligned} \text{Q.8} \quad & \int (ax^2 + bx + c) \cdot dx \\ &= a \int x^2 \cdot dx + b \int x \cdot dx + c \int 1 \cdot dx \\ &= a \frac{x^3}{3} + b \frac{x^2}{2} + cx + C_1 \\ & \quad \uparrow \\ & \quad \text{Int. Constant} \end{aligned}$$

$$\begin{aligned} \text{Q.9} \quad & \int (2x^2 + e^x) \cdot dx \\ &= 2 \int x^2 \cdot dx + \int e^x \cdot dx \\ &= 2 \frac{x^3}{3} + e^x + C \end{aligned}$$

$$\begin{aligned} \text{Q.10} \quad & \int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \cdot dx \\ &= \int \left(x + \frac{1}{x} - 2\right) \cdot dx \\ &= \frac{x^2}{2} + \log|x| - 2x + C \end{aligned}$$

$$\begin{aligned} \text{Q.11} \quad & \int \frac{x^3 + 5x^2 - 4}{x^2} \cdot dx \\ &= \int (x + 5 - 4x^{-2}) \cdot dx \\ &= \frac{x^2}{2} + 5x - 4 \frac{x^{-1}}{-1} + C \\ &= \frac{x^2}{2} + 5x + \frac{4}{x} + C \end{aligned}$$

$$\begin{aligned} \text{Q.12} \quad & \int \frac{x^3 + 3x + 4}{\sqrt{x}} \cdot dx \\ &= \int \left(x^{\frac{5}{2}} + 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}\right) \cdot dx \\ &= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 3 \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 4 \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \end{aligned}$$

$$\begin{aligned} \text{Q.13} \quad & \int \frac{x^3 - x^2 + x - 1}{(x-1)} \cdot dx \\ &= \int \frac{x^2(x-1) + 1(x-1)}{(x-1)} \cdot dx \\ &= \int \frac{\cancel{(x-1)}(x^2+1)}{\cancel{(x-1)}} \cdot dx = \left(\frac{x^3}{3} + x + C\right) \end{aligned}$$

$$\begin{aligned} &= \frac{2}{7} x^{\frac{7}{2}} + 2 x^{\frac{3}{2}} + 8 x^{\frac{1}{2}} + C \\ &= \frac{2}{7} x^3 \sqrt{x} + 2 x \sqrt{x} + 8 \sqrt{x} + C \end{aligned}$$

$$\boxed{Q.14} \int (1-x)\sqrt{x} \cdot dx$$

$$= \int (x^{1/2} - x^{3/2}) \cdot dx$$

$$= \frac{x^{3/2}}{(3/2)} - \frac{x^{5/2}}{(5/2)} + C$$

$$= \frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} + C$$

$$\boxed{Q.15} \int \sqrt{x} (3x^2 + 2x + 3) \cdot dx$$

$$= \int (3x^{5/2} + 2x^{3/2} + 3x^{1/2}) \cdot dx$$

$$= 3x \frac{x^{7/2}}{(7/2)} + 2 \frac{x^{5/2}}{(5/2)} + 3 \frac{x^{3/2}}{(3/2)} + C$$

$$= \frac{6}{7} x^{7/2} + \frac{4}{5} x^{5/2} + 2x^{3/2} + C$$

$$\boxed{Q.16} \int (2x - 3\cos x + e^x) \cdot dx$$

$$= 2 \int x \cdot dx - 3 \int \cos x \cdot dx + \int e^x \cdot dx$$

$$= 2 \left(\frac{x^2}{2} \right) - 3(\sin x) + e^x + C$$

$$\boxed{Q.17} \int (2x^2 - 3\sin x + 5\sqrt{x}) \cdot dx$$

$$= 2 \frac{x^3}{3} - 3(-\cos x) + 5 \frac{x^{3/2}}{3/2} + C$$

$$= \frac{2x^3}{3} + 3\cos x + \frac{10x^{3/2}}{3} + C$$

$$\boxed{Q.18} \int \sec x (\sec x + \tan x) \cdot dx$$

$$= \int (\sec^2 x + \sec x \cdot \tan x) \cdot dx$$

$$= \tan x + \sec x + C$$

$$\boxed{Q.20} \int \frac{2 - 3\sin x}{\cos^2 x} \cdot dx$$

$$= \int (2\sec^2 x - 3 \tan x \cdot \sec x) \cdot dx$$

$$= 2 \tan x - 3 \sec x + C$$

$$\boxed{Q.19} \int \frac{\sec^2 x}{\csc^2 x} \cdot dx$$

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} \cdot dx = \int \frac{\sin^2 x}{\cos^2 x} \cdot dx$$

$$= \int \tan^2 x \cdot dx$$

$$= \int (\sec^2 x - 1) \cdot dx$$

$$= \tan x - x + C$$

[Q.21] The anti derivative of $(\sqrt{x} + \frac{1}{\sqrt{x}})$ equals -

- (A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + c$ (B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^{\frac{1}{2}} + c$
 (C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$ (D) $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + c$

$$\int (\sqrt{x} + \frac{1}{\sqrt{x}}) \cdot dx = \int (x^{\frac{1}{2}} + x^{-\frac{1}{2}}) \cdot dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$$

[Q.22] If $\frac{d(f(x))}{dx} = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$.

- Then $f(x)$ is - (A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$ (B) $x^3 + \frac{1}{x^4} + \frac{129}{8}$
 (C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$ (D) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

Given $f'(x) = 4x^3 - \frac{3}{x^4}$
 by integrating w.r.t. 'x'

$$\int f'(x) \cdot dx = \int (4x^3 - 3x^{-4}) \cdot dx$$

$$\Rightarrow f(x) = \frac{4x^4}{4} - \frac{3x^{-3}}{(-3)} + c$$

$$\Rightarrow f(x) = x^4 + \frac{1}{x^3} + c$$

Family of curves.

Given $f(2) = 0$

$$\Rightarrow 2^4 + \frac{1}{2^3} + c = 0$$

$$\Rightarrow 16 + \frac{1}{8} + c = 0$$

$$\Rightarrow \frac{129}{8} + c = 0$$

$$\Rightarrow c = -\frac{129}{8}$$

$$f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$$

METHODS OF INTEGRATION

- Int. by Substitution
- Int. using Partial Fraction.
- Int. by Parts

Integration by Substitution

[प्रतिस्थापन द्वारा समाकलन]

$$\int \sec x \cdot dx \quad \int \csc x \cdot dx$$

$$\int \tan x \cdot dx \quad \int \cot x \cdot dx$$

$$\int f(x) \cdot dx$$

★ $\int f(g(x)) \cdot dx$ $\xrightarrow{\text{Substitution}}$ $\int h(t) \cdot dt$

$\underbrace{g(x) = t}_{\text{Substitution}}$

$$\int f(x) \cdot dx$$

$$\xrightarrow[\substack{\text{Subs.} \\ x = g(t) \\ \text{diff.} \\ dx = g'(t) \cdot dt}]{\text{Substitution}} \int f(g(t)) g'(t) \cdot dt$$

Generally, उसी function को 't' माना जाता है जिसका differentiation पहले से question में मौजूद हो।

e.g. $I = \int \frac{\sin(\tan^{-1} x)}{1+x^2} \cdot dx$

$\tan^{-1} x = t$ (Substitution)

$$\frac{1}{1+x^2} \cdot dx = dt$$

$$I = \int \sin t \cdot dt$$

$$I = -\cos t + C$$

$$I = -\cos(\tan^{-1} x) + C$$

e.g. $\int (ax+b)^n \cdot \underline{dx}$

let $(ax+b) = t$

$\Rightarrow a \cdot \underline{dx} = dt$

$\Rightarrow \underline{dx} = \frac{dt}{a}$

Substitute :

$I = \int t^n \cdot \frac{dt}{a}$

$I = \frac{1}{a} \int t^n \cdot dt$

$I = \frac{1}{a} \left(\frac{t^{n+1}}{n+1} \right) + C$

$\frac{d((ax+b)^n)}{dx} = \underline{a} \cdot \underline{n} (ax+b)^{n-1}$

$I = \frac{(ax+b)^{n+1}}{\underline{a}(n+1)} + C$

e.g. $\int \underline{\sin Kx} \cdot \underline{dx} = -\frac{\cos Kx}{K} + C$

$\int \tan x \cdot dx = \underline{\log |\sec x| + C} = -\log |\cos x| + C$

$\int \cot x \cdot dx = \underline{\log |\sin x| + C} = -\log |\operatorname{cosec} x| + C$

$\int \sec x \cdot dx = \underline{\log |\sec x + \tan x| + C} = -\log |\sec x - \tan x| + C$

$\int \operatorname{cosec} x \cdot dx = \underline{\log |\operatorname{cosec} x - \cot x| + C} = -\log |\operatorname{cosec} x + \cot x| + C$

REMEMBER

$\log\left(\frac{1}{a}\right) = -\log(a)$

$\sec^2 x - \tan^2 x = 1$

$(\underline{\sec x + \tan x}) \cdot (\underline{\sec x - \tan x}) = 1$



Proof: $\int \tan x \cdot dx = \log |\sec x| + C$

LHS = I = $\int \tan x \cdot dx \Rightarrow I = \int \frac{\sin x}{\cos x} dx$

let $\cos x = t$

$\Rightarrow -\sin x \cdot dx = dt$

$\Rightarrow \sin x \cdot dx = -dt$

Substitution

$I = \int \frac{-dt}{t} =$

$I = -\int \frac{dt}{t} = -\log |t| + C = -\log |\cos x| + C$

$I = \log |\sec x| + C$

Proof: $\int \sec x \cdot dx = \log |\sec x + \tan x| + C$

$I = \int \sec x \cdot dx$

$I = \int \sec x \cdot \frac{(\sec x + \tan x)}{(\sec x + \tan x)} \cdot dx$

$I = \int \frac{(\sec^2 x + \tan x \cdot \sec x)}{\sec x + \tan x} \cdot dx$

let $\sec x + \tan x = t$

$\Rightarrow (\sec x \cdot \tan x + \sec^2 x) dx = dt$

Substitution

$I = \int \frac{dt}{t}$

$I = \log |t| + C$

$I = \log |\sec x + \tan x| + C$

e.g. $I = \int \sin^3 x \cdot \cos^2 x \cdot dx$

$$I = \int \sin^2 x \cdot \cos^2 x \cdot \sin x \cdot dx$$

$$I = \int (1 - \cos^2 x) \cdot \cos^2 x \cdot \sin x \cdot dx$$

$$I = \int (\cos^2 x - \cos^4 x) \cdot \sin x \cdot dx$$

$$\cos x = t$$

$$\Rightarrow -\sin x \cdot dx = dt$$

$$\Rightarrow \sin x \cdot dx = -dt$$

Substitution

$$I = \int (t^2 - t^4) \cdot (-dt)$$

$$I = -\left[\frac{t^3}{3} - \frac{t^5}{5}\right] + C$$

$$I = \frac{t^5}{5} - \frac{t^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

e.g. $I = \int \frac{\cos x}{\cos(x+a)} \cdot dx$

$$\text{let } (x+a) = t$$

$$\Rightarrow dx = dt$$

Substitution,

$$I = \int \frac{\cos(t-a)}{\cos t} \cdot dt$$

$$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$$

$$I = \int \left(\frac{\cos t \cdot \cos a + \sin t \cdot \sin a}{\cos t} \right) \cdot dt$$

$$I = \int (\cos a + \tan t \cdot \sin a) \cdot dt$$

$$I = \cos a \int \frac{dt}{\cos t} + \sin a \int \tan t \cdot dt$$

$$I = \cos a (t) + \sin a \log |\sec t| + C$$

$$I = (x+a) \cdot \cos a + \sin a \cdot$$

$$\log |\sec(x+a)| + C$$

Exercise 7.2

Integrate the functions

[Q.1] $\frac{2x}{1+x^2}$

$$I = \int \frac{2x}{1+x^2} \cdot dx$$

let $1+x^2 = t$ (Substitution)

$$\Rightarrow 2x \cdot dx = dt$$

$$I = \int \frac{dt}{t} = \log|t| + C = \log|1+x^2| + C$$

[Q.2] $\frac{(\log x)^2}{x}$

$$I = \int \frac{(\log x)^2}{x} \cdot dx$$

Substitution, let $\log x = t$

$$\Rightarrow \frac{1}{x} \cdot dx = dt$$

$$I = \int t^2 \cdot dt = \frac{t^3}{3} + C = \frac{(\log x)^3}{3} + C$$

[Q.3] $I = \int \frac{1}{x+x \log x} \cdot dx$

$$I = \int \frac{1}{x(1+\log x)} \cdot dx$$

Substitution, $1+\log x = t$

$$\Rightarrow \frac{1}{x} \cdot dx = dt$$

$$I = \int \frac{dt}{t}$$

$$I = \log|t| + C$$

$$I = \log|1+\log x| + C$$

Generally इस function को 't' मान लेते हैं, जिसका differentiation 'dx' के साथ मौजूद हो



$$\boxed{\text{Q.4}} \quad I = \int \sin x \cdot \sin(\cos x) \cdot dx$$

Substitution

$$\cos x = t$$

$$\Rightarrow -\sin x \cdot dx = dt$$

$$\Rightarrow \sin x \cdot dx = -dt$$

$$I = -\int \sin t \cdot dt$$

$$I = -(-\cos t) + C$$

$$\boxed{I = \cos(\cos x) + C}$$

$$\boxed{\text{Q.5}} \quad I = \int \sin(ax+b) \cdot \cos(ax+b) \cdot dx \quad \times \frac{2}{2}$$

$$I = \frac{1}{2} \int \sin 2(ax+b) \cdot dx$$

$$\text{let } 2(ax+b) = t$$

$$\Rightarrow 2(a \cdot dx) = dt$$

$$\Rightarrow dx = \frac{dt}{2a}$$

$$I = \frac{1}{2} \times \frac{1}{2a} \int \sin t \cdot dt$$

$$I = \frac{1}{4a} (-\cos t) + C$$

$$I = -\frac{1}{4a} \cos 2(ax+b) + C$$

$$\boxed{\text{Q.6}} \quad I = \int \sqrt{ax+b} \cdot dx$$

$$\text{let } ax+b = t$$

$$\Rightarrow a \cdot dx = dt$$

$$\Rightarrow dx = \frac{1}{a} \cdot dt$$

$$I = \int \sqrt{t} \cdot \frac{1}{a} \cdot dt = \frac{1}{a} \int \sqrt{t} \cdot dt$$

$$I = \frac{1}{a} \int t^{1/2} \cdot dt$$

$$I = \frac{1}{a} \cdot \frac{t^{3/2}}{(3/2)} + C$$

$$\boxed{I = \frac{2}{3a} (ax+b)^{3/2} + C}$$

Q.7

$$I = \int x \sqrt{x+2} \cdot dx$$

Substitution,

$$\begin{aligned} x+2 &= t \\ \Rightarrow dx &= dt \end{aligned} \quad \left| \quad x = t-2 \right.$$

$$I = \int (t-2) \sqrt{t} \cdot dt$$

$$I = \int (t^{3/2} - 2t^{1/2}) \cdot dt$$

$$I = \frac{t^{5/2}}{(5/2)} - 2 \frac{t^{3/2}}{3/2} + C$$

$$I = \frac{2}{5} (x+2)^{5/2} - \frac{4}{3} (x+2)^{3/2} + C$$

Q.8

$$I = \int x \sqrt{1+2x^2} \cdot dx$$

$$\text{let } 1+2x^2 = t$$

$$\Rightarrow 4x \cdot dx = dt$$

$$\Rightarrow x dx = \frac{dt}{4}$$

$$I = \int \sqrt{t} \cdot \frac{dt}{4}$$

$$I = \frac{1}{4} \int t^{1/2} \cdot dt$$

$$I = \frac{1}{4} \frac{t^{3/2}}{3/2} + C$$

$$I = \frac{1}{6} (1+2x^2)^{3/2} + C$$

Q.9

$$I = \int (4x+2) \sqrt{x^2+x+1} \cdot dx$$

$$I = 2 \int (2x+1) \sqrt{x^2+x+1} \cdot dx$$

$$\text{let } x^2+x+1 = t$$

$$\Rightarrow (2x+1) \cdot dx = dt$$

$$I = 2 \int \sqrt{t} \cdot dt$$

$$I = 2 \left(\frac{2}{3} t^{3/2} \right) + C$$

$$I = \frac{4}{3} (x^2+x+1)^{3/2} + C$$

Q.10 $I = \int \frac{1}{x - \sqrt{x}} \cdot dx$

$$I = \int \frac{1}{(\sqrt{x}-1) \sqrt{x}} \cdot dx \rightarrow$$

let $\sqrt{x} - 1 = t$

$$\Rightarrow \frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow \frac{dx}{\sqrt{x}} = 2dt$$

$$I = \int \frac{1}{t} \cdot 2dt$$

$$I = 2 \int \frac{1}{t} \cdot dt$$

$$I = 2 \cdot \log|t| + C$$

$$I = 2 \log|\sqrt{x}-1| + C$$

Q.11 $I = \int \frac{x}{\sqrt{x+4}} \cdot dx$

let $x+4 = t$

$dx = dt$

$x = t-4$

$$I = \int \frac{t-4}{\sqrt{t}} \cdot dt$$

$$I = \int \left(t^{1/2} - 4t^{-1/2} \right) \cdot dt$$

$$I = \frac{t^{3/2}}{(3/2)} - 4 \frac{t^{1/2}}{(1/2)} + C$$

$$I = \frac{2}{3} (x+4)^{3/2} - 8 (x+4)^{1/2} + C$$

$$I = (x+4)^{1/2} \cdot \left[\frac{2}{3}(x+4) - 8 \right] + C$$

$$I = \sqrt{x+4} \cdot \left[\frac{2x+8-24}{3} \right] + C$$

$$I = \sqrt{x+4} \left[\frac{2x-16}{3} \right] + C$$

$$I = \frac{2}{3} \sqrt{x+4} \cdot (x-8) + C$$

Q.12

$$I = \int (x^3 - 1)^{\frac{1}{3}} \cdot x^5 \cdot dx$$

$$I = \int (x^3 - 1)^{\frac{1}{3}} \cdot \underline{x^3} \cdot \underline{x^2} \cdot dx$$

$$\text{let } x^3 - 1 = t$$

$$\Rightarrow x^3 = t + 1$$

$$\Rightarrow \underline{3x^2 \cdot dx} = dt$$

$$\Rightarrow \underline{x^2 \cdot dx} = \frac{dt}{3}$$

$$I = \int t^{\frac{1}{3}} \cdot (t+1) \cdot \frac{dt}{3}$$

$$I = \frac{1}{3} \cdot \int (t^{\frac{4}{3}} + t^{\frac{1}{3}}) \cdot dt$$

$$I = \frac{1}{3} \left[\frac{t^{\frac{7}{3}}}{\left(\frac{7}{3}\right)} + \frac{t^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} \right] + C$$

$$I = \frac{(x^3 - 1)^{\frac{7}{3}}}{7} + \frac{(x^3 - 1)^{\frac{4}{3}}}{4} + C$$

Q.13

$$I = \int \frac{x^2}{(2 + 3x^3)^3} \cdot dx$$

$$\text{let } (2 + 3x^3) = t$$

$$\Rightarrow 9 \cdot \underline{x^2 \cdot dx} = dt$$

$$\Rightarrow x^2 \cdot dx = \frac{dt}{9}$$

$$I = \frac{1}{9} \int \frac{dt}{(t)^3}$$

$$I = \frac{1}{9} \int t^{-3} \cdot dt$$

$$I = \frac{1}{9} \left(\frac{t^{-3+1}}{-3+1} \right) + C$$

$$I = \frac{(2 + 3x^3)^{-2}}{-18} + C$$

$$I = -\frac{1}{18(2 + 3x^3)^2} + C$$

Q.14

$$I = \int \frac{1}{x (\log x)^m} \cdot dx$$

$$x > 0, m \neq 1$$

$$\text{let } \log x = t$$

$$\Rightarrow \frac{1}{x} \cdot dx = dt$$

$$I = \int \frac{dt}{t^m}$$

$$I = \int t^{-m} \cdot dt$$

$$I = \frac{t^{-m+1}}{-m+1} + C$$

$$I = \frac{(\log x)^{1-m}}{1-m} + C$$

Q.15

$$I = \int \frac{x}{9-4x^2} \cdot dx$$

$$\text{let } 9-4x^2 = t$$

$$\Rightarrow -8x \cdot dx = dt$$

$$\Rightarrow \underline{x \, dx} = \frac{dt}{-8}$$

$$I = \int \frac{1}{t} \cdot \frac{dt}{-8}$$

$$I = -\frac{1}{8} \int \frac{dt}{t}$$

$$I = -\frac{1}{8} \log |t| + C$$

$$I = -\frac{1}{8} \log |9-4x^2| + C$$



Exercise 7.2

→ Integration by

Substitution

Integrate the functions

Q.16 e^{2x+3}

$$I = \int e^{2x+3} \cdot dx$$

Substitution,

$$\text{Let } 2x+3 = t$$

$$\Rightarrow 2 dx = dt$$

$$\Rightarrow dx = \frac{dt}{2}$$

$$I = \int e^t \cdot \frac{dt}{2} = \frac{1}{2} \int e^t \cdot dt$$

$$I = \frac{1}{2} e^t + c$$

$$I = \frac{1}{2} e^{2x+3} + c$$

Generally उस function को 't' मानते हैं, जिसका differentiatⁿ dx के साथ मौजूद हो

$$(ax+b) \rightarrow t$$

Q.17 $I = \int \frac{x}{e^{x^2}} \cdot dx$

$$\text{Let } x^2 = t$$

$$\Rightarrow 2x \cdot dx = dt$$

$$\Rightarrow x dx = \frac{dt}{2}$$

$$I = \frac{1}{2} \int \frac{dt}{e^t}$$

$$I = \frac{1}{2} \int e^{-t} \cdot dt$$

$$I = \frac{1}{2} \frac{e^{-t}}{-1} + c$$

$$I = -\frac{1}{2} e^{-x^2} + c$$

$$I = -\frac{1}{2e^{x^2}} + c$$

$$I = \int e^{-x^2} \cdot x \cdot dx$$

$$-x^2 = t$$

$$-2x \cdot dx = dt$$

$$I = -\frac{1}{2} \int e^t \cdot dt$$

$$-\frac{1}{2} e^t$$

Q.18 $I = \int \frac{e^{\tan^{-1}x}}{1+x^2} \cdot dx$

let $\tan^{-1}x = t$

$\Rightarrow \frac{1}{1+x^2} \cdot dx = dt$

$I = \int e^t \cdot dt$

$I = e^t + c$

$I = e^{\tan^{-1}x} + c$

Q.19 $I = \int \frac{e^{2x} - 1}{e^{2x} + 1} \cdot dx$

$I = \int \frac{e^x(e^x - e^{-x})}{e^x(e^x + e^{-x})} \cdot dx$

let $e^x + e^{-x} = t$

$\Rightarrow [e^x + e^{-x} \cdot (-1)] dx = dt$

$\Rightarrow (e^x - e^{-x}) \cdot dx = dt$

$I = \int \frac{dt}{t}$

$I = \log|t| + c$

$I = \log|e^x + e^{-x}| + c$

Q.20 $I = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \cdot dx$

let $e^{2x} + e^{-2x} = t$

$\Rightarrow (2e^{2x} - 2e^{-2x}) \cdot dx = dt$

$\Rightarrow (e^{2x} - e^{-2x}) \cdot dx = \frac{dt}{2}$

$I = \frac{1}{2} \int \frac{dt}{t}$

$I = \frac{1}{2} \log|t| + c$

$I = \frac{1}{2} \log|e^{2x} + e^{-2x}| + c$

Q.21 $I = \int \tan^2(2x-3) \cdot dx = \frac{1}{2} \int \tan^2 t \cdot dt$

let $2x-3 = t$ $\rightarrow$ $= \frac{1}{2} \int (\sec^2 t - 1) \cdot dt$

$\Rightarrow 2dx = dt$

$\Rightarrow dx = \frac{dt}{2}$ $= \frac{1}{2} (\tan t - t) + C$

$= \frac{1}{2} \left[\tan(2x-3) - (2x-3) \right] + C$

$I = \frac{1}{2} \tan(2x-3) - x + \frac{3}{2} + C$

Constant Constant

$I = \frac{1}{2} \tan(2x-3) - x + C_1$

Q.22 $I = \int \sec^2(7-4x) \cdot dx = \int \sec^2 t \cdot \frac{dt}{-4}$

$7-4x = t$ (let)

$-4dx = dt$

$I = -\frac{1}{4} (\tan t) + C$

$I = -\frac{1}{4} \tan(7-4x) + C$

Q.23 $I = \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} \cdot dx = \int t \cdot dt = \frac{t^2}{2} + C$

let $\sin^{-1} x = t$

$\Rightarrow \frac{1}{\sqrt{1-x^2}} \cdot dx = dt$

$= \frac{(\sin^{-1} x)^2}{2} + C$



Q.24

$$I = \int \frac{2 \cos x - 3 \sin x}{6 \cos x + 4 \sin x} \cdot dx = \frac{1}{2} \int \frac{2 \cos x - 3 \sin x}{3 \cos x + 2 \sin x} \cdot dx$$

$$\left. \begin{aligned} \text{let } (3 \cos x + 2 \sin x) &= t \\ \Rightarrow (-3 \sin x + 2 \cos x) dx &= dt \end{aligned} \right\} = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log |t| + C$$

$$I = \frac{1}{2} \log |3 \cos x + 2 \sin x| + C$$

Q.25

$$I = \int \frac{1}{\cos^2 x (1 - \tan x)^2} \cdot dx$$

$$I = \int \frac{\sec^2 x}{(1 - \tan x)^2} \cdot dx \rightarrow dt$$

$$\text{let } 1 - \tan x = t$$

$$\Rightarrow -\sec^2 x \cdot dx = dt$$

$$\Rightarrow \sec^2 x \cdot dx = -dt$$

$$I = \int \frac{-dt}{(t)^2}$$

$$I = - \int t^{-2} \cdot dt$$

$$I = - \left(\frac{t^{-2+1}}{-2+1} \right) + C$$

$$I = t^{-1} + C$$

$$I = \frac{1}{t} + C$$

$$I = \frac{1}{1 - \tan x} + C$$

$$\boxed{\text{Q.26}} \quad I = \int \frac{\cos \sqrt{x}}{\sqrt{x}} \cdot dx$$

$$\text{let } (\sqrt{x}) = t$$

$$\Rightarrow \frac{1}{2\sqrt{x}} \cdot dx = dt$$

$$\Rightarrow \frac{1}{\sqrt{x}} \cdot dx = 2 dt$$

$$= \int \cos \sqrt{x}$$

$$I = \int \cos t \cdot 2 dt$$

$$I = 2 \int \cos t \cdot dt$$

$$I = 2 \cdot \sin t + C$$

$$I = 2 \sin \sqrt{x} + C$$

$$\boxed{\text{Q.27}} \quad I = \int \sqrt{\sin 2x} \cdot \cos 2x \cdot dx$$

$$\text{let } \sin 2x = t$$

$$\Rightarrow \cos 2x \cdot 2 dx = dt$$

$$\Rightarrow \cos 2x \cdot dx = \frac{dt}{2}$$

$$I = \int \sqrt{t} \cdot \frac{dt}{2}$$

$$I = \frac{1}{2} \int t^{1/2} \cdot dt$$

$$I = \frac{1}{2} \cdot \frac{t^{3/2}}{(3/2)} + C$$

$$I = \frac{1}{2} \cdot \frac{2}{3} (\sin 2x)^{3/2} + C$$

$$I = \frac{1}{3} (\sin 2x)^{3/2} + C$$

Exercise 7.2

→ Integration by Substitution

Integrate the functions

Q.28 $\frac{\cos x}{\sqrt{1+\sin x}}$

$$I = \int \frac{\cos x}{\sqrt{1+\sin x}} \cdot \underline{dx}$$

let $1+\sin x = t$

$\Rightarrow \underline{\cos x \cdot dx} = dt$

Generally इस function को 't' मानते हैं, जिसका differentiation dx के साथ मेल खाएगा

$$I = \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}} \cdot dt$$

$$I = \frac{t^{\frac{1}{2}}}{(\frac{1}{2})} + c = 2(1+\sin x)^{\frac{1}{2}} + c$$

Q.29

$$I = \int \underline{\cot x \cdot \log \sin x} \cdot \underline{dx}$$

let $(\log \sin x) = t$

$\Rightarrow \frac{1}{\sin x} \cdot \cos x \cdot dx = dt$

$\Rightarrow \underline{\cot x \cdot dx} = dt$

$$I = \int t \cdot dt$$

$$I = \frac{t^2}{2} + c$$

$$I = \frac{(\log \sin x)^2}{2} + c$$

Q.30

$$I = \int \frac{\sin x}{1+\cos x} \cdot \underline{dx}$$

let $1+\cos x = t$

$\Rightarrow -\sin x \cdot dx = dt$

$\Rightarrow \sin x \cdot dx = \underline{-dt}$

$$I = - \int \frac{dt}{t}$$

$$I = - \log |t| + c$$

$$I = - \log |1+\cos x| + c$$

$$\boxed{\text{Q.31}} \quad I = \int \frac{\sin x}{(1+\cos x)^2} \cdot dx \quad \rightarrow \quad I = - \int \frac{dt}{t^2}$$

$$\text{let } (1+\cos x) = t$$

$$\Rightarrow -\sin x \cdot dx = dt$$

$$\Rightarrow \sin x dx = -dt$$

$$I = - \int t^{-2} \cdot dt$$

$$I = - \left(\frac{t^{-1}}{-1} \right) + C$$

$$I = \frac{1}{t} + C = \frac{1}{1+\cos x} + C$$

$$\boxed{\text{Q.32}} \quad I = \int \frac{1}{1+\cot x} \cdot dx$$

$$I = \int \frac{1}{\frac{1+\cos x}{\sin x}} \cdot dx = \int \frac{\sin x}{\sin x + \cos x} \cdot dx \times \frac{2}{2}$$

$$I = \frac{1}{2} \int \frac{2 \sin x}{\sin x + \cos x} \cdot dx = \frac{1}{2} \int \frac{(\sin x + \cos x) - (\cos x - \sin x)}{(\sin x + \cos x)} \cdot dx$$

$$I = \frac{1}{2} \int 1 \cdot dx - \frac{1}{2} \int \frac{\cos x - \sin x}{\sin x + \cos x} \cdot dx$$

$$I = \frac{1}{2} (x) - \frac{1}{2} I_2$$

I_2

$$\text{let } \sin x + \cos x = t$$

$$(\cos x - \sin x) \cdot dx = dt$$

$$I_2 = \int \frac{\cos x - \sin x}{\sin x + \cos x} \cdot dx$$

$$I_2 = \int \frac{dt}{t} = \log|t|$$

$$= \log|\sin x + \cos x|$$

$$\boxed{I = \frac{1}{2} (x) - \frac{1}{2} \log|\sin x + \cos x| + C}$$

$$\boxed{Q.33} \quad I = \int \frac{1}{1 - \tan x} \cdot dx = \int \frac{1}{1 - \frac{\sin x}{\cos x}} \cdot dx$$

$$I = \int \frac{\cos x}{\cos x - \sin x} \cdot dx \times \frac{2}{2}$$

$$I = \frac{1}{2} \int \frac{2 \cos x}{\cos x - \sin x} \cdot dx$$

$$I = \frac{1}{2} \int \frac{(\cos x - \sin x) - (-\sin x - \cos x)}{(\cos x - \sin x)} \cdot dx$$

$$I = \frac{1}{2} \int 1 \cdot dx - \frac{1}{2} \int \frac{-\sin x - \cos x}{\cos x - \sin x} \cdot dx$$

$$\text{let } \cos x - \sin x = t$$

$$\Rightarrow (-\sin x - \cos x) \cdot dx = -dt$$

$$I = \frac{1}{2} x - \frac{1}{2} \int \frac{dt}{t}$$

$$I = \frac{1}{2} x - \frac{1}{2} \log|t| + c = \frac{x}{2} - \frac{1}{2} \log|\cos x - \sin x| + c$$

$$\boxed{Q.34} \quad I = \int \frac{\sqrt{\tan x}}{\sin x \cdot \cos x} \cdot dx$$

divide by $\cos^2 x$ in Nr & Dr.

$$I = \int \frac{\sqrt{\tan x} / \cos^2 x}{\sin x \times \cos x / \cos^2 x} \cdot dx$$

$$I = \int \frac{\sqrt{\tan x} \cdot \sec^2 x}{\tan x} \cdot dx$$

$$I = \int \frac{\sec^2 x \cdot dx}{\sqrt{\tan x}}$$

$$\text{let } \tan x = t$$

$$\sec^2 x \cdot dx = dt$$

$$I = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2} \cdot dt$$

$$I = \frac{t^{1/2}}{1/2} + c = 2\sqrt{t} + c$$

$$I = 2\sqrt{\tan x} + c$$

$$\boxed{\text{Q.35}} \quad I = \int \frac{(1+\log x)^2}{x} \cdot dx \quad \rightarrow \quad I = \int \frac{t^2}{1} \cdot dt$$

$$\text{let } (1+\log x) = t$$

$$\Rightarrow \frac{1}{x} \cdot dx = dt$$

$$I = \frac{t^3}{3} + C$$

$$I = \frac{(1+\log x)^3}{3} + C$$

$$\boxed{\text{Q.36}} \quad I = \int \frac{(x+1)(x+\log x)^2}{x} \cdot dx$$

$$\text{let } (x+\log x) = t$$

$$\Rightarrow \left(1 + \frac{1}{x}\right) dx = dt$$

$$\Rightarrow \left(\frac{x+1}{x}\right) \cdot dx = dt$$

$$I = \int \frac{t^2}{1} \cdot dt$$

$$I = \frac{t^3}{3} + C$$

$$I = \frac{(x+\log x)^3}{3} + C$$

$$\boxed{\text{Q.37}} \quad I = \int \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8} \cdot dx$$

$$\text{let } (\tan^{-1} x^4) = t$$

$$\Rightarrow \left(\frac{1}{1+(x^4)^2}\right) \cdot 4x^3 \cdot dx = dt$$

$$\Rightarrow \frac{x^3}{1+x^8} \cdot dx = \frac{dt}{4}$$

$$I = \frac{1}{4} \int \sin(t) \cdot dt$$

$$I = \frac{1}{4} (-\cos t) + C$$

$$I = -\frac{1}{4} \cos(\tan^{-1} x^4) + C$$

Q.38

$$\int \frac{10x^9 + 10^x \log_{10} e}{x^{10} + 10^x} \cdot dx \text{ equals}$$

$$\text{let } (x^{10} + 10^x) = t$$

$$\Rightarrow (10 \cdot x^9 + 10^x \cdot \log_{10} e) \cdot dx = dt$$

$$I = \int \frac{dt}{t} = \log|t| + c$$
$$= \log|x^{10} + 10^x| + c$$

option D

Q.39 $\int \frac{1 \cdot dx}{\sin^2 x \cdot \cos^2 x}$ equals —

$$I = \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cdot \cos^2 x} \cdot dx$$

$$I = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) \cdot dx$$

$$I = \int (\sec^2 x + \csc^2 x) \cdot dx$$

$$I = \tan x - \cot x + c$$

$$1 = \sin^2 x + \cos^2 x$$

No
Substitution

Integration using Trigonometric Identities

- Trigonometric Formulas (217)
- Integration Formulas (214)
- Substitution method ✓
- Practice ?

e.g. Find the integral

(I) $I = \int \cos^2 x \cdot dx$ (Degree = 2) → (Degree = 1)

Here. $\boxed{\cos^2 x = \frac{1 + \cos 2x}{2}}$

$$I = \int \frac{1 + \cos 2x}{2} \cdot dx = \frac{1}{2} \int (1 + \cos 2x) \cdot dx$$

$$I = \frac{1}{2} \int 1 \cdot dx + \frac{1}{2} \int \cos 2x \cdot dx$$

$$I = \frac{1}{2} (x) + \frac{1}{2} \left(\frac{\sin 2x}{2} \right) + C$$

$$\boxed{I = \frac{x}{2} + \frac{\sin 2x}{4} + C}$$

$\frac{a^n + b}{\sin 2x} \rightarrow \cos 2x$

(II) $I = \int \sin^3 x \cdot dx$ (I-method)

$$\left(\begin{array}{l} \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta \\ \sin^3 \theta = \frac{3 \sin \theta - \sin 3\theta}{4} \end{array} \right)$$

$$I = \int \frac{3 \sin x - \sin 3x}{4} \cdot dx$$

$$I = \frac{3}{4} \int \sin x \cdot dx - \frac{1}{4} \int \sin 3x \cdot dx$$

$$I = \frac{3}{4} (-\cos x) - \frac{1}{4} \left(-\frac{\cos 3x}{3} \right) + C$$

$$I = -\frac{3}{4} \cos x + \frac{\cos 3x}{12} + C$$

II- method

$$I = \int \sin^3 x \cdot \underline{dx} \quad (\text{Substitution})$$

$$I = \int \underline{\sin^2 x} \cdot \underline{\sin x \cdot dx}$$

$$I = \int (1 - \cos^2 x) \cdot \underline{\sin x \cdot dx} \quad \rightarrow \quad I = -\int (1 - t^2) \cdot dt$$

$$\text{Let } \cos x = t$$

$$\Rightarrow -\sin x \cdot dx = dt$$

$$\Rightarrow \underline{\sin x \cdot dx} = -dt$$

$$I = -\left(t - \frac{t^3}{3}\right) + C$$

$$I = \frac{t^3}{3} - t + C$$

$$\boxed{I = \frac{\cos^3 x}{3} - \cos x + C}$$

$$\textcircled{\text{III}} \quad I = \int \underline{\sin 3x} \cdot \underline{\sin 5x} \cdot dx \times \frac{2}{2}$$

$$I = \frac{1}{2} \int \underline{2 \sin 3x \cdot \sin 5x} \cdot dx \quad (2 \sin A \cdot \sin B)$$

$$I = \frac{1}{2} \int [\cos(3x - 5x) - \cos(3x + 5x)] \cdot dx$$

$$I = \frac{1}{2} \int (\cos 2x - \cos 8x) \cdot dx$$

$$I = \frac{1}{2} \left(\frac{\sin 2x}{2} - \frac{\sin 8x}{8} \right) + C$$

$$I = \frac{\sin 2x}{4} - \frac{\sin 8x}{16} + C$$

Exercise 7.3

Find the integrals

Q.1

$$I = \int \sin^2(2x+5) \cdot dx$$

$$\sin^2(2x+5) = \frac{1 - \cos 2(2x+5)}{2}$$

$$= \frac{1}{2} - \frac{\cos(4x+10)}{2}$$

$$I = \int \left(\frac{1}{2} - \frac{\cos(4x+10)}{2} \right) \cdot dx$$

$$I = \frac{1}{2} \int dx - \frac{1}{2} \int \cos(4x+10) \cdot dx$$

$$I = \frac{1}{2}(x) - \frac{1}{2} \frac{\sin(4x+10)}{4} + C$$

$$I = \frac{x}{2} - \frac{1}{8} \sin(4x+10) + C$$

Int. using Trigonometry

- Trigo. Formulas $\frac{2\pi}{4}$
- Int. Formulas $\frac{2\pi}{4}$
- Substitution method ✓
- Practice ✓

$$\int \cos x \cdot dx \rightarrow \sin x + C$$

Q.2 $I = \frac{2}{2} \int \sin 3x \cdot \cos 4x \cdot dx$

$$I = \frac{1}{2} \int 2 \sin 3x \cdot \cos 4x \cdot dx$$

$$2 \sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$$

$$I = \frac{1}{2} \int (\sin 7x + \sin(-x)) \cdot dx$$

$$I = \frac{1}{2} \int (\sin 7x - \sin x) \cdot dx \rightarrow I = \frac{1}{2} \left(-\frac{\cos 7x}{7} + \cos x \right) + C$$

Q.3 $I = \int \cos 2x \cdot \cos 4x \cdot \cos 6x \cdot dx \times \frac{2}{2}$

$$I = \frac{1}{2} \int (2 \cos 2x \cdot \cos 4x) \cdot \cos 6x \cdot dx$$

$$I = \frac{1}{2} \int (\cos(6x) + \cos(2x)) \cdot \cos 6x \cdot dx$$

$\cos(0) = \cos \theta$

$$I = \frac{1}{2} \int (\cos^2 6x + \cos 2x \cdot \cos 6x) \cdot dx \times \frac{2}{2}$$

$$I = \frac{1}{4} \int (2 \cos^2 6x + 2 \cos 2x \cdot \cos 6x) \cdot dx$$

$1 + \cos 12x$ $\cos 8x$

$$I = \frac{1}{4} \int [(1 + \cos 12x) + \cos 8x + \cos(4x)] \cdot dx$$

$$I = \frac{1}{4} \left(x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4} \right) + c$$

Q.4 $I = \int \sin^3(2x+1) \cdot dx$

$$I = \int \sin^2(2x+1) \cdot \sin(2x+1) \cdot dx$$

$$I = \int [1 - \cos^2(2x+1)] \cdot \sin(2x+1) \cdot dx$$

let $\cos(2x+1) = t$ ✓

$$\Rightarrow -\sin(2x+1) \cdot 2 dx = dt$$

$$\Rightarrow \sin(2x+1) \cdot dx = \left(-\frac{dt}{2}\right) \checkmark$$

$$I = -\frac{1}{2} \int (1-t^2) \cdot dt$$

$$I = -\frac{1}{2} \left(t - \frac{t^3}{3} \right) + c$$

$$I = -\frac{\cos(2x+1)}{2}$$

$$+ \frac{\cos^3(2x+1)}{6} + c$$

Q.5 $I = \int \sin^3 x \cdot \cos^3 x \cdot dx$

$\int \sin^m x \cdot \cos^n x \cdot dx$

$I = \int \sin^2 x \cdot \cos^3 x \cdot \sin x \cdot dx$

$I = \int (1 - \cos^2 x) \cdot \cos^3 x \cdot \sin x \cdot dx$

$I = \int (\cos^3 x - \cos^5 x) \cdot \sin x \cdot dx$

$\cos x = t$
 $-\sin x \cdot dx = dt$

$I = -\left(\frac{t^4}{4} - \frac{t^6}{6}\right) + C$

$I = \frac{t^6}{6} - \frac{t^4}{4} + C$

$I = -\int (t^3 - t^5) \cdot dt$

$I = \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$

Q.6 $I = \int (\sin x \cdot \sin 2x \cdot \sin 3x) \cdot dx \times \frac{2}{2}$

$I = \frac{1}{2} \int (2 \sin x \cdot \sin 2x) \cdot \sin 3x \cdot dx$

$2 \sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$

$I = \frac{1}{2} \int (\cos x - \cos 3x) \cdot \sin 3x \cdot dx$

$I = \frac{1}{2} \times \frac{1}{2} \int (2 \cos x \cdot \sin 3x - 2 \cos 3x \cdot \sin 3x) \cdot dx$

$I = \frac{1}{4} \int (\sin(4x) + \sin(2x)) - (\sin 6x) \cdot dx$

$I = \frac{1}{4} \left(-\frac{\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6} \right) + C$

Q.7

$$I = \frac{1}{2} \int 2 \sin 4x \cdot \sin 8x \cdot dx$$

$$I = \frac{1}{2} \int (\cos 4x - \cos 12x) \cdot dx$$

$$I = \frac{1}{2} \left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12} \right] + C \quad \checkmark$$

Q.8

$$I = \int \frac{1 - \cos x}{1 + \cos x} \cdot dx$$

$$\rightarrow I = \int \frac{\cancel{2} \sin^2 \frac{x}{2}}{\cancel{2} \cos^2 \frac{x}{2}}$$

$$2 \cos^2 \theta = 1 + \cos 2\theta$$

$$2 \sin^2 \theta = 1 - \cos 2\theta$$

$$I = \int \tan^2 \frac{x}{2} \cdot dx$$

$$I = \int \left[\sec^2 \left(\frac{x}{2} \right) - 1 \right] \cdot dx$$

$$I = \frac{\tan \frac{x}{2}}{\left(\frac{1}{2} \right)} - x + C = \left(2 \tan \frac{x}{2} - x + C \right)$$

Q.9

$$I = \int \frac{\cos x}{1 + \cos x} \cdot dx$$

$$\Rightarrow I = \int \frac{2 \cos^2 \frac{x}{2} - 1}{2 \cos^2 \frac{x}{2}} \cdot dx$$

$$1 + \cos x = 2 \cos^2 \frac{x}{2}$$

$$\cos x = 2 \cos^2 \frac{x}{2} - 1$$

$$\Rightarrow I = \int \left(1 - \frac{1}{2} \sec^2 \frac{x}{2} \right) \cdot dx$$

$$I = x - \frac{1}{2} \left[\frac{\tan \frac{x}{2}}{\left(\frac{1}{2} \right)} \right] + C$$

$$I = x - \tan \frac{x}{2} + C$$

Exercise 7.3

Int. using Trigonometry

Q.10 $I = \int \sin^4 x \cdot dx$

$$I = \int (\sin^2 x)^2 \cdot dx$$

$$I = \int \left(\frac{1 - \cos 2x}{2} \right)^2 \cdot dx$$

$$I = \frac{1}{4} \int (1 + \cos^2 2x - 2\cos 2x) \cdot dx$$

$$I = \frac{1}{4} \int \left(\frac{1 - 2\cos 2x}{1} + \frac{1 + \cos 4x}{2} \right) \cdot dx$$

$$I = \frac{1}{4} \int \frac{3 - 4\cos 2x + \cos 4x}{2} \cdot dx$$

$$I = \frac{1}{4} \times \frac{1}{2} \left(3x - \frac{\sin 2x}{2} + \frac{\sin 4x}{4} \right) + C$$

$$I = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

Q.11 $I = \int \cos^4 2x \cdot dx = \int (\cos^2 2x)^2 \cdot dx$

$$I = \int \left(\frac{1 + \cos 4x}{2} \right)^2 \cdot dx = \frac{1}{4} \int (1 + 2\cos 4x + \cos^2 4x) \cdot dx$$

$$I = \frac{1}{4} \int \left(1 + 2\cos 4x + \frac{1 + \cos 8x}{2} \right) \cdot dx \times \frac{2}{2}$$

$$I = \frac{1}{8} \int (3 + 4 \cos 4x + \cos 8x) \cdot dx$$

$$I = \frac{1}{8} \left(3x + \cancel{4} \frac{\sin 4x}{\cancel{4}} + \frac{\sin 8x}{8} \right) + C$$

$$\boxed{I = \frac{3x}{8} + \frac{\sin 4x}{8} + \frac{\sin 8x}{64} + C}$$

$$\boxed{\text{Q.12}} \quad I = \int \frac{\sin^2 x}{1 + \cos x} \cdot dx = \int \frac{1 - \cos^2 x}{(1 + \cos x)} \cdot dx$$

$$I = \int \frac{(1 + \cancel{\cos x}) \cdot (1 - \cos x)}{(1 + \cancel{\cos x})} \cdot dx = \underline{x - \sin x + C}$$

$$\boxed{\text{Q.13}} \quad I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} \cdot dx$$

$$I = \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{(\cos x - \cos \alpha)} \cdot dx = 2 \int \frac{(\cos^2 x - \cos^2 \alpha)}{\cos x - \cos \alpha} \cdot dx$$

$$I = 2 \int \frac{(\cancel{\cos x} - \cos \alpha)(\cos x + \cos \alpha)}{(\cancel{\cos x} - \cos \alpha)} \cdot dx$$

$$\boxed{I = 2 (\sin x + x \cos \alpha) + C}$$

Q.14 $I = \int \frac{\cos x - \sin x}{1 + \sin 2x} \cdot dx$

$$I = \int \frac{\cos x - \sin x}{1 + 2 \sin x \cos x} \cdot dx = \int \frac{\cos x - \sin x}{\sin^2 x + \cos^2 x + 2 \sin x \cos x} \cdot dx$$

$$I = \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2} \cdot dx$$

Let $(\sin x + \cos x) = t$

$$\Rightarrow (\cos x - \sin x) \cdot dx = dt$$

$$I = \int \frac{dt}{t^2}$$

$$I = \int t^{-2} \cdot dt$$

$$I = -\frac{1}{t} + C$$

$$I = \frac{t^{-1}}{-1} + C$$

$$I = -\frac{1}{\sin x + \cos x} + C$$

Q.15 $I = \int \tan^3 2x \cdot \sec 2x \cdot dx$

$$I = \int \tan^2 2x \cdot \tan 2x \cdot \sec 2x \cdot dx$$

$$I = \int (\sec^2 2x - 1) \cdot \sec 2x \cdot \tan 2x \cdot dx$$

Let $\sec 2x = t$

$$\Rightarrow 2 \cdot \sec 2x \cdot \tan 2x \cdot dx = dt$$

$$\Rightarrow \sec 2x \cdot \tan 2x \cdot dx = \frac{dt}{2}$$

$$I = \frac{1}{2} \int (t^2 - 1) \cdot dt$$

$$I = \frac{1}{2} \left(\frac{t^3}{3} - t \right) + C$$

$$I = \frac{\sec^3 2x}{6} - \frac{\sec 2x}{2} + C$$

Exercise 7.3

→ Int. using Trigonometry

Find the integrals

Q.16 $I = \int \tan^4 x \cdot dx$

$$I = \int \tan^2 x \cdot \tan^2 x \cdot dx$$

$$I = \int \tan^2 x \cdot (\sec^2 x - 1) \cdot dx = \int \tan^2 x \cdot \sec^2 x \cdot dx - \int \tan^2 x \cdot dx$$

$$I = \underbrace{\int \tan^2 x \cdot \sec^2 x \cdot dx}_{I_1} - \underbrace{\int (\sec^2 x - 1) \cdot dx}_{I_2}$$

$$I_1 = \int \tan^2 x \cdot \sec^2 x \cdot dx$$

Let $\tan x = t$

$$\sec^2 x \cdot dx = dt$$

$$\Rightarrow I_1 = \int t^2 dt$$

$$I_1 = \frac{t^3}{3} + C'$$

$$I_1 = \frac{\tan^3 x}{3} + C'$$

$$I = \frac{\tan^3 x}{3} - (\tan x - x) + C$$

$$I = \frac{\tan^3 x}{3} - \tan x + x + C$$

$$\boxed{\text{Q.17}} \quad I = \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cdot \cos^2 x} \cdot dx$$

$$I = \int \left(\frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x} \right) \cdot dx$$

$$I = \int (\tan x \cdot \sec x + \cot x \cdot \operatorname{cosec} x) \cdot dx$$

$$I = \sec x - \operatorname{cosec} x + C$$

$$\boxed{\text{Q.18}} \quad I = \int \frac{(\cos 2x) + 2\sin^2 x}{\cos^2 x} \cdot dx$$

$$I = \int \frac{(1 - 2\sin^2 x) + 2\sin^2 x}{\cos^2 x} \cdot dx = \int \sec^2 x \cdot dx$$

$$I = \tan x + C$$

$$\boxed{\text{Q.19}} \quad I = \int \frac{1}{\sin x \cdot \cos^3 x} \cdot dx = \int \frac{(\sec^3 x / \cos x)}{(\sin x / \cos x)} \cdot dx$$

$$I = \int \frac{\sec^4 x}{\tan x} \cdot dx$$

$$I = \int \frac{\sec^2 x \cdot \sec^2 x}{\tan x} \cdot dx$$

$$I = \int \left(\frac{1 + \tan^2 x}{\tan x} \right) \cdot \sec^2 x \cdot dx$$

$$\tan x = t$$

$$\sec^2 x \cdot dx = dt$$

$$\rightarrow I = \int \frac{1+t^2}{t} \cdot dt$$

$$I = \int \left(\frac{1}{t} + t \right) \cdot dt$$

$$I = \log |t| + \frac{t^2}{2} + C$$

$$I = \log |\tan x| + \frac{\tan^2 x}{2} + C$$

$$\boxed{\text{Q.20}} \quad I = \int \frac{\cos 2x}{(\cos x + \sin x)^2} \cdot dx$$

$$I = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} \cdot dx = \int \frac{(\cancel{\cos x + \sin x}) \cdot (\cos x - \sin x)}{(\cancel{\cos x + \sin x})^2} \cdot dx$$

$$I = \int \frac{(\cos x - \sin x)}{(\cos x + \sin x)} \cdot dx$$

$$\text{let } (\cos x + \sin x) = t$$

$$\Rightarrow (-\sin x + \cos x) \cdot dx = dt$$

$$\begin{aligned} &\rightarrow I = \int \frac{dt}{t} \\ &I = \log |t| + C \\ &\boxed{I = \log |\cos x + \sin x| + C} \end{aligned}$$

$$\boxed{\text{Q.21}} \quad I = \int \sin^{-1}(\cos x) \cdot dx$$

$$I = \int \sin^{-1}(\sin(90^\circ - x)) \cdot dx$$

$$I = \int \left(\frac{\pi}{2} - x \right) \cdot dx$$

$$90^\circ = \frac{\pi}{2}$$

$$\boxed{I = \frac{\pi}{2} x - \frac{x^2}{2} + C}$$

Q.22
★

$$I = \int \frac{1}{\cos(x-a) \cdot \cos(x-b)} \cdot dx$$

Sin(a-b) multiply
Divide

$\sin(a-b) \leftarrow \frac{1}{\sin(x-a) \cdot \sin(x-b)}$
 $\cos(a-b) \leftarrow \frac{1}{\sin(x-a) \cdot \cos(x-b)}$

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a) \cdot \cos(x-b)} \cdot dx$$

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin(\overset{\text{Sin(A-B) = SinA \cdot CosB - CosA \cdot SinB}}{(x-b) - (x-a)})}{\cos(x-a) \cdot \cos(x-b)} \cdot dx$$

$$I = \frac{1}{\sin(a-b)} \int \frac{[\sin(x-b) \cdot \cos(x-a) - \cos(x-b) \cdot \sin(x-a)]}{\cos(x-a) \cdot \cos(x-b)} dx$$

$$I = \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)}{\cos(x-b)} - \frac{\sin(x-a)}{\cos(x-a)} \right] \cdot dx$$

$$I = \frac{1}{\sin(a-b)} \int [\tan(x-b) - \tan(x-a)] \cdot dx$$

$$I = \frac{1}{\sin(a-b)} \cdot [\log |\sec(x-b)| - \log |\sec(x-a)|] + C$$

$$I = \frac{1}{\sin(a-b)} \cdot \log \left| \frac{\sec(x-b)}{\sec(x-a)} \right| + C$$

$$I = \frac{1}{\sin(a-b)} \cdot \log \left| \frac{\cos(x-a)}{\cos(x-b)} \right| + C$$

Q.23 $I = \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cdot \cos^2 x} \cdot dx$ is equal to —

$$I = \int (\sec^2 x - \operatorname{cosec}^2 x) \cdot dx$$

~~$$I = \int \sec^2 x - \operatorname{cosec}^2 x \cdot dx$$~~

$$I = \tan x - (-\cot x) + C$$

$$I = \tan x + \cot x + C \quad \text{Option (A)}$$

Q.24 $I = \int \frac{e^x (1+x)}{\cos^2(e^x \cdot x)} \cdot dx$ equals —

Let $e^x \cdot x = t$

$$\Rightarrow (e^x \cdot x + e^x \cdot 1) dx = dt$$

$$\Rightarrow e^x (x+1) \cdot dx = dt$$

$$I = \int \frac{dt}{\cos^2(t)}$$

$$I = \int \sec^2 t \cdot dt$$

$$I = \tan(t) + C$$

$$I = \tan(e^x \cdot x) + C$$

Integration of the Type - $\frac{1}{Q}$, $\frac{1}{\sqrt{Q}}$, $\frac{L}{Q}$, $\frac{L}{\sqrt{Q}}$

Q $\rightarrow$ Quadratic (द्विघात)

L $\rightarrow$ Linear (रेखिक)

Standard Formulas

$$\int \frac{1}{Q} \cdot dx$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{Q}} \cdot dx$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + c$$

Hints for Proofs

(in general)

$(a^2 - x^2) \rightarrow$ Put $x = a \sin \theta$

$(a^2 + x^2) \rightarrow$ Put $x = a \tan \theta$

$(x^2 - a^2) \rightarrow$ Put $x = a \sec \theta$

Proof (I) $I = \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

Put $x = a \sec \theta$ ✓
(Substitution)

$$(dx = a \sec \theta \cdot \tan \theta \cdot d\theta)$$

$$I = \int \frac{a \sec \theta \cdot \tan \theta \cdot d\theta}{a^2 \sec^2 \theta - a^2}$$

$$I = \frac{1}{a} \int \frac{\sec \theta \cdot \tan \theta \cdot d\theta}{\tan^2 \theta}$$

$$I = \frac{1}{a} \int \frac{(\frac{1}{\cos \theta})}{(\frac{\sin \theta}{\cos \theta})} \cdot d\theta$$

$$I = \frac{1}{a} \int \csc \theta \cdot d\theta$$

$$I = \frac{1}{a} \int \operatorname{cosec} \theta \cdot d\theta$$

$$I = \frac{1}{a} \log |\operatorname{cosec} \theta - \cot \theta| + C$$

$$I = \frac{1}{a} \log \left| \frac{x}{\sqrt{x^2 - a^2}} - \frac{a}{\sqrt{x^2 - a^2}} \right| + C$$

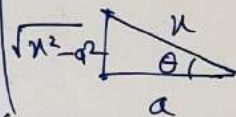
$$I = \frac{1}{a} \log \left| \frac{x-a}{\sqrt{x^2 - a^2}} \right| + C$$

$$I = \frac{1}{a} \log \left| \frac{\cancel{(x-a)} \sqrt{x-a}}{\sqrt{\cancel{(x-a)}(x+a)}} \right| + C$$

$$I = \frac{1}{a} \log \left| \sqrt{\frac{x-a}{x+a}} \right| + C = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$x = a \sec \theta$$

$$\sec \theta = \frac{x}{a} = \frac{H}{B}$$



$$\operatorname{cosec} \theta = \frac{H}{P} = \frac{x}{\sqrt{x^2 - a^2}}$$

$$\cot \theta = \frac{B}{P} = \frac{a}{\sqrt{x^2 - a^2}}$$

Proof (ii) $I = \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + C$

$$I = \int \frac{dx}{\sqrt{a^2 - x^2}}$$

Put $x = a \sin \theta$ ✓
Substitution,

$$dx = a \cos \theta \cdot d\theta \quad \checkmark$$

$$I = \int \frac{a \cos \theta \cdot d\theta}{\sqrt{a^2 - a^2 \sin^2 \theta}}$$

$$I = \int \frac{a \cos \theta \cdot d\theta}{a \sqrt{1 - \sin^2 \theta}}$$

$$I = \int \frac{\cos \theta}{\cos \theta} \cdot d\theta$$

$$I = \int 1 \cdot d\theta$$

$$I = \theta + C$$

$$I = \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\begin{aligned} x &= a \sin \theta \\ \frac{x}{a} &= \sin \theta \\ \sin^{-1} \left(\frac{x}{a} \right) &= \theta \end{aligned}$$

e.g.

$$\boxed{\frac{1}{Q}, \frac{1}{\sqrt{Q}}}$$

e.g. $I = \int \frac{dx}{x^2 - 16}$

$$I = \int \frac{dx}{x^2 - 4^2} \quad (a=4)$$

$$I = \frac{1}{2 \times 4} \log \left| \frac{x-4}{x+4} \right| + c$$

$$I = \int \frac{dx}{\sqrt{3+2x-x^2}}$$

(Completing the square method)

$$\begin{aligned} & \cancel{I} = \int \frac{dx}{\sqrt{3+2x-x^2}} \\ & = 3 - (x^2 - 2x) \\ & = 3 - (x^2 - 2x \cdot 1 + 1^2 - 1^2) \\ & = 3 - [(x-1)^2] + 1 \\ & = 4 - (x-1)^2 \end{aligned}$$

$$I = \int \frac{dx}{\sqrt{3+2x-x^2}} = \int \frac{dx}{\sqrt{4-(x-1)^2}} = \int \frac{dx}{\sqrt{2^2-(x-1)^2}}$$

Let $x-1 = t$

$\underline{dx = dt}$

$$I = \int \frac{dt}{\sqrt{2^2 - t^2}} = \sin^{-1} \frac{t}{a} + c$$

$(a=2)$

$$\cancel{I} = \boxed{I = \sin^{-1} \left(\frac{x-1}{2} \right) + c}$$

Examples on $\int \frac{L}{Q}$ and $\int \frac{L}{\sqrt{Q}}$

$L = \text{linear} = Px + Q$

$Q = \text{Quadratic} = ax^2 + bx + c$

$$\frac{Px+Q}{ax^2+bx+c}, \quad \frac{Px+Q}{\sqrt{ax^2+bx+c}}$$

$$\boxed{L = A(Q') + B}$$

$(A, B \in \mathbb{R})$

$$\boxed{(Px+Q) = A \frac{d(ax^2+bx+c)}{dx} + B}$$

$A \& B = ?$

Comparison

e.g. $I = \int \frac{x+2}{2x^2+6x+5} \cdot dx$

$\frac{L}{Q}$

Let $L = A \cdot Q' + B$ $A, B = ?$

$$x+2 = A \frac{d(2x^2+6x+5)}{dx} + B$$

$$x+2 = A(4x+6) + B$$

$$x+2 = 4Ax + 6A + B$$

by comparison.

$(x) \quad 1 = 4A \Rightarrow A = \frac{1}{4}$

$(\text{Const}) \quad 2 = 6A + B$

$$\Rightarrow 2 = 6\left(\frac{1}{4}\right) + B$$

$$\Rightarrow 2 = \frac{3}{2} + B$$

$$\Rightarrow \frac{1}{2} = B$$

$$x+2 = \frac{1}{4}(4x+6) + \frac{1}{2}$$

Replace

$$I = \int \frac{\frac{1}{4}(4x+6) + \frac{1}{2}}{2x^2+6x+5} \cdot dx$$

$$I = \frac{1}{4} \int \frac{4x+6}{2x^2+6x+5} \cdot dx \rightarrow I_1$$

$$+ \frac{1}{2} \int \frac{1}{2x^2+6x+5} \cdot dx \rightarrow I_2$$

$$I_1 = \int \frac{4x+6}{2x^2+6x+5} \cdot dx$$

$$2x^2+6x+5 = t$$

$$4x+6 \cdot dx = dt$$

$$I_1 = \int \frac{dt}{t} = \log|t| + c_1$$

$$I_1 = \log|2x^2+6x+5| + c_1$$

$$I_2 = \int \frac{1}{2x^2+6x+5} \cdot dx$$

$$I_2 = \int \frac{1}{2(x^2+3x)+5} \cdot dx$$

$$I_2 = \int \frac{1}{2\left(x^2+3x+\frac{9}{4}-\frac{9}{4}\right)+5} \cdot dx$$

$$I_2 = \int \frac{dx}{2\left(x+\frac{3}{2}\right)^2 + \frac{1}{2}}$$

$$I_2 = \frac{1}{2} \int \frac{du}{\left(u+\frac{3}{2}\right)^2 + \frac{1}{4}}$$

$\frac{L}{Q}$

$$I_2 = \frac{1}{2} \int \frac{dx}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

Formula

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$a = \frac{1}{2}$$

$$I_2 = \frac{1}{2} \left[\frac{1}{\left(\frac{1}{2}\right)} \cdot \tan^{-1} \left(\frac{x + \frac{3}{2}}{\frac{1}{2}} \right) \right] + C_2$$

$$I_2 = \tan^{-1}(2x + 3) + C_2$$

$$I = \frac{1}{4} \textcircled{I_1} + \frac{1}{2} \textcircled{I_2}$$

$$I = \frac{1}{4} \log |2x^2 + 6x + 5| + \frac{1}{2} \tan^{-1}(2x + 3) + C$$

$$\frac{1}{Q}, \frac{1}{\sqrt{Q}}$$

Short

$$\frac{L}{Q}, \frac{L}{\sqrt{Q}}$$

lengthy

Exercise 7.4

First - Remember the formulas

Type-I Q1 to Q9 Today

Type-II Q10 to Q15

Type-III Q.16 to Q.23

~~Q.1~~ Integrate the functions.

$$\boxed{\text{Q.1}} \quad I = \int \frac{3x^2}{x^6 + 1} \cdot dx = \int \frac{3x^2}{(x^3)^2 + 1} \cdot dx$$

$$\text{Let } x^3 = t \quad \Rightarrow \quad 3x^2 \cdot dx = dt$$
$$I = \int \frac{dt}{t^2 + 1} = \tan^{-1}(t) + c$$

$$\boxed{I = \tan^{-1}(x^3) + c}$$

$$\boxed{\text{Q.2}} \quad I = \int \frac{dx}{\sqrt{1+4x^2}} = \int \frac{dx}{\sqrt{4\left(\frac{1}{4} + x^2\right)}} = \frac{1}{2} \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 + x^2}}$$

$$I = \frac{1}{2} \log \left| x + \sqrt{\left(\frac{1}{2}\right)^2 + x^2} \right| + c$$

$$I = \frac{1}{2} \log \left| x + \sqrt{\frac{1}{4} + x^2} \right| + c$$

$$I = \frac{1}{2} \log \left| x + \sqrt{\frac{1+4x^2}{4}} \right| + c$$

$$I = \frac{1}{2} \cdot \log \left| \frac{2x + \sqrt{1+4x^2}}{2} \right| + c$$

$$I = \frac{1}{2} \cdot \log |2x + \sqrt{1+4x^2}| - \frac{1}{2} \log 2 + c$$

$$\boxed{I = \frac{1}{2} \log |2x + \sqrt{1+4x^2}| + c_1}$$

[Q.3]

$$I = \int \frac{1}{\sqrt{(2-x)^2 + 1}} \cdot dx$$

$$\text{let } (2-x) = t$$

$$\Rightarrow -dx = dt$$

$$I = - \int \frac{dt}{\sqrt{t^2 + 1}} = - \log |t + \sqrt{t^2 + 1}| + C$$

$$I = \log \left| \frac{1}{(2-x) + \sqrt{(2-x)^2 + 1}} \right| + C$$

$(2-x)^2 = 4 + x^2 - 4x$
(+1)

$$I = \log \left| \frac{1}{2-x + \sqrt{5+x^2-4x}} \right| + C$$

[Q.4]

$$I = \int \frac{dx}{\sqrt{9-25x^2}} = \int \frac{dx}{\sqrt{25\left(\frac{9}{25} - x^2\right)}}$$

$$I = \frac{1}{5} \int \frac{dx}{\sqrt{\left(\frac{3}{5}\right)^2 - x^2}} = \frac{1}{5} \cdot \sin^{-1}\left(\frac{x}{\left(\frac{3}{5}\right)}\right) + C$$

$$I = \frac{1}{5} \sin^{-1}\left(\frac{5x}{3}\right) + C$$

[Q.5] $I = \int \frac{3x}{1+2x^4} \cdot dx = \int \frac{3x}{1+2(x^2)^2} \cdot dx$

$$\text{let } x^2 = t$$

$$\Rightarrow 2x \cdot dx = dt$$

$$\Rightarrow x \cdot dx = \frac{dt}{2}$$

$$I = \int \frac{\frac{3}{2} dt}{1+2t^2} = \frac{3}{2 \times 2} \int \frac{dt}{\frac{1}{2} + t^2}$$

$$I = \frac{3}{4} \int \frac{dt}{\left(\frac{1}{\sqrt{2}}\right)^2 + t^2} = \frac{3}{4} \cdot \frac{1}{\left(\frac{1}{\sqrt{2}}\right)} \cdot \tan^{-1}\left(\frac{t}{\frac{1}{\sqrt{2}}}\right) + C$$

$$I = \frac{3\sqrt{2}}{4} \cdot \tan^{-1}(t\sqrt{2}) + C = \frac{3}{2\sqrt{2}} \cdot \tan^{-1}(\sqrt{2}x^2) + C$$

Q.6 $I = \int \frac{x^2}{1-x^6} \cdot dx \Rightarrow I = \int \frac{x^2}{1-(x^3)^2} \cdot dx$

Let $x^3 = t$ ✓

$\Rightarrow 3x^2 \cdot dx = dt$

$\Rightarrow x^2 \cdot dx = \frac{dt}{3}$ ✓

$I = \frac{1}{3} \int \frac{dt}{1-t^2}$

$(a=1)$

$I = \frac{1}{3} \cdot \frac{1}{2} \log \left| \frac{1+t}{1-t} \right| + C$

$I = \frac{1}{6} \log \left| \frac{1+x^3}{1-x^3} \right| + C$

Q.7 $(I) = \int \frac{x-1}{\sqrt{x^2-1}} \cdot dx = \underbrace{\int \frac{x}{\sqrt{x^2-1}} \cdot dx}_{I_1} - \underbrace{\int \frac{1}{\sqrt{x^2-1}} \cdot dx}_{I_2}$

$I = I_1 - I_2$

Formula

$I_1 = \int \frac{x}{\sqrt{x^2-1}} \cdot dx$

Let $x^2-1 = t$

$\Rightarrow 2x \cdot dx = dt$

$\Rightarrow x \cdot dx = \frac{dt}{2}$

$I_1 = \frac{1}{2} \int \frac{dt}{\sqrt{t}} = \frac{1}{2} \int t^{-\frac{1}{2}} \cdot dt$

$I_1 = \frac{1}{2} \cdot \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C_1$

$I_1 = \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C_1 = \sqrt{x^2-1} + C_1$

$I_2 = \log |x + \sqrt{x^2-1}| + C_2$

$I = I_1 - I_2$

$I = \sqrt{x^2-1} + C_1 - \log |x + \sqrt{x^2-1}| + C_2$

$I = \sqrt{x^2-1} - \log |x + \sqrt{x^2-1}| + C$

$C = C_1 - C_2$

$$\boxed{\text{Q.8}} \quad I = \int \frac{x^2}{\sqrt{x^6 + a^6}} \cdot dx = \int \frac{x^2}{\sqrt{(x^3)^2 + a^6}} \cdot dx$$

~~Let~~ let $x^3 = t$ ✓

$$\Rightarrow 3x^2 \cdot dx = dt$$

$$\Rightarrow \underline{x^2 dx} = \frac{dt}{3} \quad \checkmark$$

$$I = \frac{1}{3} \int \frac{dt}{\sqrt{t^2 + (a^3)^2}}$$

$$I = \frac{1}{3} \log \left| \underline{t} + \sqrt{\underline{t^2} + a^6} \right| + c$$

$$\boxed{I = \frac{1}{3} \log \left| x^3 + \sqrt{x^6 + a^6} \right| + c}$$

$$\boxed{\text{Q.9}} \quad I = \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} \cdot dx$$

let $\tan x = t$

$$\underline{\sec^2 x \cdot dx} = dt$$

$$I = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

$$I = \log \left| \underline{t} + \sqrt{\underline{t^2} + 4} \right| + c$$

$$\boxed{I = \log \left| \tan x + \sqrt{\tan^2 x + 4} \right| + c}$$

Exercise 7.4

Type-I Q1 to Q9

Type-II Q10 to Q15

Today

Type-III Q16 to Q23

Completing the square method 10th

Q.11 $I = \int \frac{-1}{9x^2 + 6x + 5} \cdot dx$

$$I = \frac{1}{9} \int \frac{1}{x^2 + \frac{2x}{3} + \frac{5}{9}} \cdot dx = \frac{1}{9} \int \frac{1}{\underbrace{x^2 + \frac{2x}{3} + \left(\frac{1}{3}\right)^2}_{\left(x + \frac{1}{3}\right)^2} - \underbrace{\left(\frac{1}{3}\right)^2 + \frac{5}{9}}_{\frac{4}{9}}} \cdot dx$$

$$I = \frac{1}{9} \int \frac{1}{\left(x + \frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2} \cdot dx = \frac{1}{9} \int \frac{1}{\left(x + \frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2} \cdot dx$$

$$= \frac{1}{9} \times \frac{1}{\left(\frac{2}{3}\right)} \cdot \tan^{-1}\left(\frac{x + \frac{1}{3}}{\frac{2}{3}}\right) + c$$

$$I = \frac{1}{6} \tan^{-1}\left(\frac{3x+1}{2}\right) + c$$

Q.12 $I = \int \frac{1}{\sqrt{7-6x-x^2}} \cdot dx$

Similarly Q.14

$$I = \int \frac{1}{\sqrt{7 - (x^2 + 6x + 3^2 - 3^2)}} \cdot dx = \int \frac{1}{\sqrt{7+9 - (x+3)^2}} \cdot dx$$

$$I = \int \frac{1}{\sqrt{4^2 - (x+3)^2}} \cdot dx = \sin^{-1}\left(\frac{x+3}{4}\right) + c$$

Q.15

$$I = \int \frac{1}{\sqrt{(x-a)(x-b)}} \cdot dx$$

Similarly Q 10 & Q 13

$$I = \int \frac{1}{\sqrt{x^2 - (a+b)x + ab}} \cdot dx$$

$$I = \int \frac{1}{\sqrt{x^2 - (a+b)x + \left(\frac{a+b}{2}\right)^2 - \left(\frac{a+b}{2}\right)^2 + ab}} \cdot dx$$

$$I = \int \frac{1}{\sqrt{\left[x - \frac{a+b}{2}\right]^2 - \left(\frac{a^2+b^2}{2}\right)}} \cdot dx$$

$$I = \int \frac{1}{\sqrt{\left(x - \frac{a+b}{2}\right)^2 - \left(\sqrt{\frac{a^2+b^2}{2}}\right)^2}} \cdot dx$$

$$I = \log \left| \left(x - \frac{a+b}{2}\right) + \sqrt{\left(x - \frac{a+b}{2}\right)^2 - \left(\sqrt{\frac{a^2+b^2}{2}}\right)^2} \right| + C$$

$$I = \log \left| \left(x - \frac{a+b}{2}\right) + \sqrt{(x-a)(x-b)} \right| + C$$



Exercise 7.4

Type-I Q1 - Q9 ✓

Type-II Q10 - Q15 ✓

Type-III Q16 - Q23

Today

$$\left(\frac{L}{Q}, \frac{L}{\sqrt{Q}} \right)$$

Completing
the square method

Q.16

$$I = \int \frac{4x+1}{\sqrt{2x^2+x-3}} \cdot dx \rightarrow dt$$

$$I = \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}} \cdot dt$$

$$\begin{aligned} 2x^2 + x - 3 &= t \text{ (let)} \\ (4x+1) dx &= dt \end{aligned}$$

$$I = \frac{t^{\frac{1}{2}}}{(\frac{1}{2})} + c = 2\sqrt{t} + c = 2\sqrt{2x^2+x-3} + c$$

Q.17

$$I = \int \frac{x+2}{\sqrt{x^2-1}} \cdot dx \quad \left(\frac{L}{\sqrt{Q}} \right)$$

$$I = \underbrace{\frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} \cdot dx}_{I_1} + 2 \underbrace{\int \frac{1}{\sqrt{x^2-1}} \cdot dx}_{I_2} = \frac{1}{2} \textcircled{I_1} + 2 \textcircled{I_2}$$

$$I_1 = \int \frac{2x}{\sqrt{x^2-1}} \cdot dx$$

$$\text{let } (x^2-1) = t$$

$$\Rightarrow 2x dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}}$$

$$= 2\sqrt{t} + c_1 = 2\sqrt{x^2-1} + c_1$$

$$I_2 = \log|x + \sqrt{x^2-1}| + c_2$$



$$I = \frac{1}{2} \textcircled{I_1} + 2 \textcircled{I_2}$$

$$\boxed{2\sqrt{x^2-1} + C_1}$$

$$\boxed{\log |x + \sqrt{x^2-1}| + C_2}$$

$$I = \underbrace{\sqrt{x^2-1} + \frac{C_1}{2}} + 2 \underbrace{\log |x + \sqrt{x^2-1}| + C_2}$$

$$I = \sqrt{x^2-1} + 2 \log |x + \sqrt{x^2-1}| + C$$

$$\textcircled{C = \frac{C_1}{2} + 2C_2}$$

Q.18

$$I = \int \frac{5x-2}{1+2x+3x^2} \cdot dx$$

$$\left(\frac{L}{Q} \right) \rightarrow \frac{Ax + B}{Q}$$

$$5x-2 = A \cdot \frac{d(1+2x+3x^2)}{dx} + B$$

$$\Rightarrow 5x-2 = \underline{A(2+6x)} + \underline{B}$$

By comparison.

$$\textcircled{x \rightarrow} 5 = A \cdot 6$$

$$\Rightarrow \boxed{A = \frac{5}{6}}$$

Constant $\rightarrow$

$$-2 = 2A + B$$

$$\boxed{B = -\frac{11}{3}}$$

$$I = \int \frac{\textcircled{5x-2}}{1+2x+3x^2} \cdot dx$$

$$I = \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{\textcircled{3x^2+2x+1}} \cdot dx$$

$$I = \frac{5}{6} \underbrace{\int \frac{2+6x}{3x^2+2x+1} \cdot dx}_{\textcircled{I_1}} - \frac{11}{3} \underbrace{\int \frac{1}{3x^2+2x+1} \cdot dx}_{\textcircled{I_2}}$$

$$\boxed{I = \frac{5}{6} \cdot \textcircled{I_1} - \frac{11}{3} \cdot I_2}$$

$$I_1 = \int \frac{2+6x}{3x^2+2x+1} \cdot dx \rightarrow dt$$

$$\text{let } 3x^2+2x+1=t$$

$$\Rightarrow (6x+2) \cdot dx = dt$$

$$I_1 = \int \frac{dt}{t} = \log|t| + C_1$$

$$I_1 = \log|3x^2+2x+1| + C_1$$

$$I_2 = \int \frac{1}{3x^2+2x+1} \cdot dx$$

By completing square method

$$I_2 = \frac{1}{3} \int \frac{1}{x^2 + \frac{2x}{3} + \frac{1}{3}} \cdot dx = \frac{1}{3} \int \frac{1 \cdot dx}{x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2 + \frac{1}{3}}$$

$$I_2 = \frac{1}{3} \int \frac{dx}{\left(x + \frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2}$$

$$\frac{2}{9} = \left(\frac{\sqrt{2}}{3}\right)^2$$

$$I_2 = \frac{1}{3} \cdot \left(\frac{1}{\frac{\sqrt{2}}{3}}\right) \cdot \tan^{-1}\left(\frac{x + \frac{1}{3}}{\frac{\sqrt{2}}{3}}\right) + C_2$$

$$I_2 = \frac{1}{\sqrt{2}} \cdot \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C_2$$

$$I = \frac{5}{6} (I_1) - \frac{11}{3} (I_2)$$

$$I = \frac{5}{6} \left\{ \log|3x^2+2x+1| + C_1 \right\} - \frac{11}{3} \left\{ \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C_2 \right\}$$

$$I = \frac{5}{6} \log|3x^2+2x+1| - \frac{11}{3\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

$$\left(\frac{5}{6} C_1 - \frac{11}{3} C_2 \right)$$

$$\boxed{\text{Q.19}} \quad I = \int \frac{6x+7}{\sqrt{(x-5)(x-4)}} \cdot dx = \int \frac{6x+7}{\sqrt{x^2-9x+20}} \cdot dx$$

$$6x+7 = A \cdot \frac{d(x^2-9x+20)}{dx} + B$$

$$\Rightarrow \underline{6x+7} = A(2x-9) + B = \underline{3(2x-9) + 34}$$

$$(x) \rightarrow 6 = 2A \Rightarrow \boxed{A=3}$$

$$\text{Constant} \rightarrow 7 = -9A + B \Rightarrow 7 = -27 + B \Rightarrow \boxed{B=34}$$

$$I = \int \frac{6x+7}{\sqrt{x^2-9x+20}} \cdot dx = \int \frac{3(2x-9) + 34}{\sqrt{x^2-9x+20}} \cdot dx$$

$$I = 3 \underbrace{\int \frac{(2x-9)}{\sqrt{x^2-9x+20}} \cdot dx}_{(I_1)} + 34 \underbrace{\int \frac{1}{\sqrt{x^2-9x+20}} \cdot dx}_{(I_2)}$$

$$\boxed{I = 3(I_1) + 34(I_2)}$$

$$I_1 = \int \frac{(2x-9)}{\sqrt{x^2-9x+20}} \cdot dx \rightarrow dt = \int \frac{1}{\sqrt{t}} \cdot dt$$

$$\underline{I_1} = 2\sqrt{t} + C, = \underline{2\sqrt{x^2-9x+20} + C}$$

$$I_2 = \int \frac{1}{\sqrt{x^2 - 9x + 20}} \cdot dx \quad \text{Complete Square}$$

$$I_2 = \int \frac{1}{\sqrt{x^2 - 9x + \left(\frac{9}{2}\right)^2 + 20 - \left(\frac{9}{2}\right)^2}} \cdot dx$$

$$I_2 = \int \frac{dx}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} \quad \left(-\frac{1}{4}\right) = -\left(\frac{1}{2}\right)^2$$

$$I_2 = \log \left| \left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20} \right| + C_2$$

$$I = 3(I_1) + 34(I_2)$$

$$I = 3 \left\{ 2\sqrt{x^2 - 9x + 20} + C_1 \right\} + 34 \left\{ \log \left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right| + C_2 \right\}$$

$$I = 6\sqrt{x^2 - 9x + 20} + 34 \log \left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right| + C$$

$$C = 3C_1 + 34C_2$$

Q.20 $I = \int \frac{x+2}{\sqrt{4x-x^2}} \cdot dx$

$$\frac{C}{\sqrt{Q}}$$

$$(x+2) = A \cdot \frac{d(4x-x^2)}{dx} + B$$

Comparison.

$$(x+2) = A(4-2x) + B$$

$$(x) \rightarrow 1 = -2A \Rightarrow A = -\frac{1}{2}$$

$$\text{Constant} \rightarrow 2 = 4A + B \Rightarrow 2 = -2 + B$$

$$B = 4$$

$$\left(-\frac{1}{2}\right)$$

$$(4)$$

$$I = \int \frac{x+2}{\sqrt{4x-x^2}} \cdot dx = \int \frac{-\frac{1}{2}(4-2x) + 4}{\sqrt{4x-x^2}} \cdot dx$$

$$I = -\frac{1}{2} \underbrace{\int \frac{4-2x}{\sqrt{4x-x^2}} \cdot dx}_{I_1} + 4 \underbrace{\int \frac{1}{\sqrt{4x-x^2}} \cdot dx}_{I_2}$$

$$I = -\frac{1}{2}(I_1) + 4(I_2)$$

$$I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}} \cdot dx \xrightarrow{\substack{4-2x \rightarrow dt \\ 4x-x^2 \rightarrow t}} dt = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C_1$$

$$I_1 = 2\sqrt{4x-x^2} + C_1$$

$$I_2 = \int \frac{1}{\sqrt{4x-x^2}} \cdot dx$$

By Completing the Square method

$$I_2 = \int \frac{dx}{\sqrt{-(x^2-4x+2^2-2^2)}} = \int \frac{dx}{\sqrt{-(x-2)^2+4}}$$

$$I_2 = \int \frac{dx}{\sqrt{2^2-(x-2)^2}} = \sin^{-1}\left(\frac{x-2}{2}\right) + C_2$$

$$I = -\frac{1}{2}(I_1) + 4(I_2)$$

$$I = -\frac{1}{2} \left(2\sqrt{4x-x^2} + C_1 \right) + 4 \left[\left(\sin^{-1}\left(\frac{x-2}{2}\right) \right) + C_2 \right]$$

$$I = -\sqrt{4x-x^2} + 4 \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Q.24 $I = \int \frac{dx}{x^2+2x+2}$ equals —

$$(x+1)^2 = x^2+2x+1$$

$$I = \int \frac{dx}{\underbrace{x^2+2x+1}_{(x+1)^2} + 1} = \int \frac{dx}{(x+1)^2 + 1^2}$$

option - B

$$I = \frac{1}{1} \tan^{-1}\left(\frac{x+1}{1}\right) + C = \underline{\underline{\tan^{-1}(x+1) + C}}$$

Q.25 $I = \int \frac{dx}{\sqrt{9x-4x^2}}$ equals

Complete Square

$$I = \int \frac{dx}{\sqrt{-4\left(x^2 - \frac{9x}{4}\right)}}$$

$$I = \int \frac{dx}{\sqrt{-4\left[x^2 - \frac{9x}{4} + \left(\frac{9}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right]}}$$

$$I = \int \frac{dx}{\sqrt{4 \times \left(\frac{9}{8}\right)^2 - 4\left(x - \frac{9}{8}\right)^2}} = \frac{1}{2} \int \frac{dx}{\sqrt{\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2}}$$

$$I = \frac{1}{2} \sin^{-1}\left(\frac{x - \frac{9}{8}}{\frac{9}{8}}\right) + C$$

$$I = \frac{1}{2} \sin^{-1}\left(\frac{8x-9}{9}\right) + C$$

option B

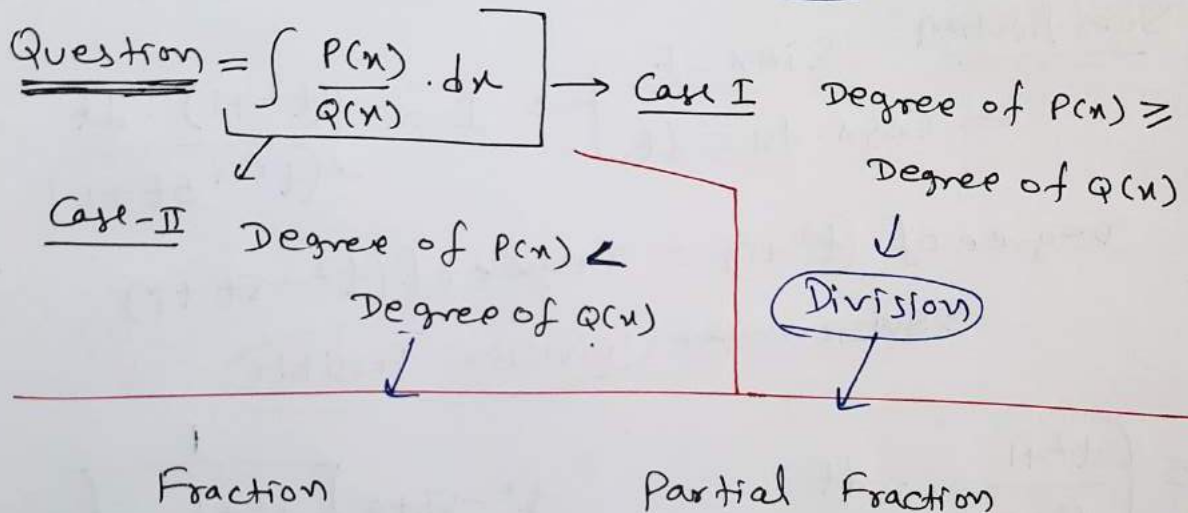
Integration by Partial Fraction [आंशिक भिन्न द्वारा समाकलन]

$$\frac{2}{x+1} - \frac{3}{x+2} = \frac{2x+4-3x-3}{(x+1)(x+2)} = \frac{-x+1}{x^2+3x+2} \quad \left(\frac{L}{Q}\right)$$

$\uparrow \quad \uparrow$
 Partial Fraction (आंशिक भिन्न)

$\uparrow$
 Question

$\uparrow$
 Fraction (भिन्न)



① $\frac{Px+q}{(x-a)(x-b)} \rightarrow \frac{A}{(x-a)} + \frac{B}{(x-b)}$ ✓

② $\frac{Px^2+qx+r}{(x-a)(x-b)(x-c)} \rightarrow \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$ ✓

③ $\frac{Px+q}{(x-a)^2} \rightarrow \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$

④ $\frac{Px^2+qx+r}{(x-a)^2(x-b)} \rightarrow \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$

⑤ $\frac{Px^2+qx+r}{(x-a)(x^2+bx+c)} \rightarrow \frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$

No roots

Note $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

$$\int \frac{1}{(ax+b)} \cdot dx = \frac{\log|ax+b|}{a} + C$$

$$\int \frac{1}{x} \cdot dx = \log|x| + C$$

e.g. Integrate $I = \int \frac{(\sin^2 x + 1) \cdot \cos x}{(\sin^2 x - 5 \sin x + 6)} \cdot dx$

Substitution $\sin x = t$
 $\Rightarrow \cos x \cdot dx = dt$ } $I = \int \frac{(t^2 + 1) \cdot dt}{(t^2 - 5t + 6)}$

Degree of $(t^2 + 1)$ = Degree of $(t^2 - 5t + 6)$

~~Division~~ Division Possible

$$I = \int \frac{t^2 + 1}{t^2 - 5t + 6} \cdot dt$$

$$\begin{array}{r} 1 \\ t^2 - 5t + 6 \overline{) t^2 + 1} \\ \underline{t^2 - 5t + 6} \\ 5t - 5 \end{array}$$

$$I = \int \left[1 + \frac{5t - 5}{(t^2 - 5t + 6)} \right] \cdot dt$$

$$I = \int \left[1 + \frac{5t - 5}{(t-2)(t-3)} \right] \cdot dt$$

Fraction $\frac{5t - 5}{(t-2)(t-3)} = \frac{A}{t-2} + \frac{B}{t-3}$

$\xrightarrow{-5} \xrightarrow{10}$

$$\frac{5t-5}{(t-2)(t-3)} = \frac{A}{t-2} + \frac{B}{t-3}$$

Proper method $5t-5 = A(t-3) + B(t-2)$

$$5t-5 = At + Bt - 3A - 2B$$

Comparison.

t $\rightarrow 5 = A + B$ — (1)

Constant $\rightarrow -5 = -3A - 2B$ — (2)

$$\begin{array}{r|l} 5 = 3A + 2B \\ 10 = 2A + 2B \\ \hline -5 = A \end{array} \quad \left| \begin{array}{l} 5 = A + B \\ -5 \\ \hline B = 10 \end{array} \right.$$

Shortcut.

$$\frac{5t-5}{(t-2)(t-3)} = \frac{A}{t-2} + \frac{B}{t-3}$$

$\times \quad \times$

$t-2=0$
 $t=2$

$$A = \frac{5(2)-5}{(2)-3} = \frac{5}{-1} = -5$$

$$B = \frac{5(3)-5}{(3)-2} = \frac{10}{1} = 10$$

$$I = \int \left[1 + \frac{5t-5}{(t-2)(t-3)} \right] \cdot dt = \int \left[1 + \frac{-5}{(t-2)} + \frac{10}{(t-3)} \right] \cdot dt$$

$$I = t - 5 \log|t-2| + 10 \log|t-3| + C$$

$t = \sin x$

$$I = \sin x - 5 \log|\sin x - 2| + 10 \log|\sin x - 3| + C$$

e.g. Integrate $I = \int \frac{3x-2}{(x+1)^2(x+3)} \cdot dx$

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+3)}$$

Proper method

$$\Rightarrow (3x-2) = A(x+1)(x+3) + B(x+3) + C(x+1)^2$$

$$\Rightarrow (3x-2) = A(x^2+4x+3) + B(x+3) + C(x^2+2x+1)$$

$$(x^2) \rightarrow 0 = A + C \quad \text{--- (1)}$$

$$(x) \rightarrow 3 = 4A + B + 2C \quad \text{--- (2)}$$

$$\text{Constant} \rightarrow -2 = 3A + 3B + C \quad \text{--- (3)}$$

$$\left. \begin{array}{l} A = \frac{11}{4} \\ B = -\frac{5}{2} \\ C = -\frac{11}{4} \end{array} \right\}$$

$$I = \int \frac{(3x-2)}{(x+1)^2(x+3)} \cdot dx = \int \left(\frac{\left(\frac{11}{4}\right)}{(x+1)} + \frac{\left(-\frac{5}{2}\right)}{(x+1)^2} + \frac{\left(-\frac{11}{4}\right)}{(x+3)} \right) \cdot dx$$

$$I = \frac{11}{4} \int \frac{dx}{x+1} - \frac{5}{2} \int \frac{dx}{(x+1)^2} - \frac{11}{4} \int \frac{1}{(x+3)} \cdot dx$$

$$I = \frac{11}{4} \log|x+1| - \frac{5}{2} \left(\frac{-1}{(x+1)} \right) - \frac{11}{4} \log|x+3| + C$$

$$I = \frac{11}{4} \log|x+1| + \frac{5}{2(x+1)} - \frac{11}{4} \log|x+3| + C$$

Exercise 7.5 → Int. by Partial Fraction.

Q.1 $I = \int \frac{x}{(x+1)(x+2)} \cdot dx$

Degree of $x = 1$ (N_r)

Degree of $(x+1)(x+2) = 2$ (D_r)

$$\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

Proper method

$$x = A(x+2) + B(x+1)$$

Comparison.

(1)

$$1 = A + B \quad \text{--- (1)}$$

Constant

$$0 = 2A + B \quad \text{--- (2)}$$

Shortcut

$$1 = -A \Rightarrow \boxed{A = -1} \quad \checkmark$$

$$\boxed{B = 2} \quad \checkmark$$

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$x+1=0 \Rightarrow x=-1$ (for A)
 $x+2=0 \Rightarrow x=-2$ (for B)

$$A = \frac{-1}{-1+2} = \frac{-1}{1} = -1 \quad \Bigg| \quad B = \frac{-2}{-2+1} = \frac{-2}{-1} = 2$$

$$I = \int \frac{x}{(x+1)(x+2)} \cdot dx$$

$$I = \int \left[\frac{-1}{(x+1)} + \frac{2}{(x+2)} \right] \cdot dx$$

$$I = -1 \cdot \log|x+1| + 2 \log|x+2| + C$$

$$I = \log \frac{1}{|x+1|} + \log|x+2|^2 + C$$

$$\boxed{I = \log \left| \frac{(x+2)^2}{x+1} \right| + C}$$

Q. 2 $I = \int \frac{dx}{x^2-9}$

$$\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$$

$$\Rightarrow 1 = A(x+3) + B(x-3)$$

Comparison. $(x) \rightarrow 0 = A + B \quad \text{--- (1)}$

Const. $\rightarrow 1 = 3A - 3B \quad \text{--- (2)}$

$$\begin{bmatrix} A = \frac{1}{6} \\ B = -\frac{1}{6} \end{bmatrix}$$

$$I = \int \frac{1}{x^2-9} \cdot dx = \int \left[\frac{\frac{1}{6}}{x-3} + \frac{-\frac{1}{6}}{x+3} \right] \cdot dx$$

$$I = \frac{1}{6} \log|x-3| - \frac{1}{6} \log|x+3| + C$$

$$I = \frac{1}{6} \log \left| \frac{x-3}{x+3} \right| + C$$

$$\begin{aligned} \log m - \log n \\ = \log \frac{m}{n} \end{aligned}$$

Q. 3 $I = \int \frac{3x-1}{(x-1)(x-2)(x-3)} \cdot dx$

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$\Rightarrow 3x-1 = A(x^2-5x+6) + B(x^2-4x+3) + C(x^2-3x+2)$$

Comparison. $(x^2) \rightarrow 0 = A + B + C$

$(x) \rightarrow 3 = -5A - 4B - 3C$

Constant $\rightarrow -1 = 6A + 3B + 2C$

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 4 \end{aligned}$$

$$I = \int \frac{3x-1}{(x-1)(x-2)(x-3)} \cdot dx$$

$$I = \int \left(\frac{1}{x-1} + \frac{-5}{x-2} + \frac{4}{x-3} \right) \cdot dx$$

$$I = \log|x-1| - 5 \cdot \log|x-2| + 4 \cdot \log|x-3| + C$$

$$\boxed{\text{Q.4}} \quad I = \int \frac{x}{(x-1)(x-2)(x-3)} \cdot dx$$

$$I = \int \left(\frac{(1/2)}{(x-1)} + \frac{(-2)}{(x-2)} + \frac{(3/2)}{(x-3)} \right) \cdot dx$$

$$I = \frac{1}{2} \log|x-1| - 2 \log|x-2| + \frac{3}{2} \log|x-3| + C$$

$$\boxed{\text{Q.5}} \quad I = \int \frac{2x}{x^2+3x+2} \cdot dx$$

$$I = \int \frac{2x}{(x+1)(x+2)} \cdot dx$$

$$\begin{aligned} & \text{Factorization of } x^2+3x+2 \\ & x^2+3x+2 = x^2+2x+x+2 \\ & = x(x+2) + 1(x+2) \\ & = (x+1)(x+2) \end{aligned}$$

$$I = \int \left[\frac{-2}{(x+1)} + \frac{4}{(x+2)} \right] \cdot dx$$

$$I = -2 \log|x+1| + 4 \log|x+2| + C$$

Q.6

$$I = \int \frac{1-x^2}{x(1-2x)} \cdot dx$$

$$\text{Deg.}(Nr) = 2$$

$$\text{Deg.}(Dr) = 2$$

$$\frac{1-x^2}{x(1-2x)} = \frac{(1-x^2)}{x-2x^2}$$

Division

$$= \frac{1}{2} + \frac{-\frac{x}{2} + 1}{x-2x^2}$$

$$= \frac{1}{2} + \frac{\left(\frac{-x+2}{2}\right)}{(x-2x^2)} \rightarrow \times$$

$$= \frac{1}{2} + \frac{(x-2)}{2(2x^2-x)} = \frac{1}{2} + \frac{(x-2)}{2x(2x-1)}$$

$$\frac{x-2}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$$

$$x = \frac{1}{2}$$

$$A = \frac{-2}{-1} = 2$$

$$B = -3$$

$$I = \int \frac{1-x^2}{x(1-2x)} \cdot dx$$

$$I = \int \left(\frac{1}{2} + \frac{1}{2} \left[\frac{x-2}{x(2x-1)} \right] \right) \cdot dx$$

$$I = \int \left[\frac{1}{2} + \frac{1}{2} \left(\frac{2}{x} - \frac{3}{2x-1} \right) \right] \cdot dx$$

$$I = \frac{x}{2} + \log|x| - \frac{3}{2} \frac{\log|2x-1|}{2} + C$$

Q.7

$$I = \int \frac{x}{(x^2+1) \cdot (x-1)} \cdot dx$$

$\downarrow$
No Factors

$$\frac{x}{(x^2+1) \cdot (x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$

$$\Rightarrow x = (Ax+B)(x-1) + C(x^2+1)$$

$$\Rightarrow 1 \cdot x = \underline{Ax^2} - \underline{Ax} + \underline{Bx} - \underline{B} + \underline{Cx^2} + \underline{C}$$

Comparison,

$$\left. \begin{array}{l} (x^2) \rightarrow 0 = A+C \\ (x) \rightarrow 1 = -A+B \\ (\text{Const.}) \rightarrow 0 = -B+C \end{array} \right\} \begin{array}{l} A = -\frac{1}{2} \\ B = \frac{1}{2} \\ C = \frac{1}{2} \end{array}$$

$$I = \int \frac{x}{(x^2+1) \cdot (x-1)} \cdot dx$$

$\rightarrow$ $Ax+B$

$$I = \int \left(\frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1} + \frac{(\frac{1}{2})}{x-1} \right) \cdot dx$$

$$I = \frac{1}{2} \left\{ \int \frac{-x+1}{x^2+1} \cdot dx + \int \frac{1}{x-1} \cdot dx \right\}$$

$$I = \frac{1}{2} \times \frac{1}{2} \int \frac{-x \cdot x^2}{\underline{x^2+1}} \cdot dx + \frac{1}{2} \int \frac{1 \cdot dx}{x^2+1} + \frac{1}{2} \int \frac{1}{x-1} \cdot dx$$

$$I = -\frac{1}{4} \int \frac{(2x \cdot dx) \rightarrow dt}{\underline{x^2+1} \rightarrow t} + \frac{1}{2} \tan^{-1}(x) + \frac{1}{2} \log|x-1| + C$$

$$I = -\frac{1}{4} \log|x^2+1| + \frac{1}{2} \tan^{-1} x + \frac{1}{2} \log|x-1| + C$$

Q.8

$$I = \int \frac{x}{(x-1)^2 \cdot (x+2)} \cdot dx$$

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

$$\Rightarrow x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$\Rightarrow x = A(x^2 + x - 2) + B(x+2) + C(x^2 - 2x + 1)$$

Comparison, $(x^2) \rightarrow 0 = A + C$ — (1)

$(x) \rightarrow 1 = A + B - 2C$ — (2)

Const. $\rightarrow 0 = -2A + 2B + C$ — (3)

$$\underline{A = \frac{1}{3}}, \quad \underline{B = \frac{1}{2}}, \quad \underline{C = -\frac{1}{3}}$$

$$I = \int \frac{x}{(x-1)^2 \cdot (x+2)} \cdot dx$$

$$I = \int \left\{ \frac{\frac{1}{3}}{(x-1)} + \frac{\frac{1}{2}}{(x-1)^2} - \frac{\frac{1}{3}}{(x+2)} \right\} \cdot dx$$

$$I = \frac{1}{3} \cdot \log|x-1| + \frac{1}{2} \left(\frac{-1}{x-1} \right) - \frac{1}{3} \log|x+2| + C$$

$$I = \left(\frac{1}{3} \right) \log \left| \frac{x-1}{x+2} \right| - \left(\frac{1}{2} \right) \left(\frac{1}{x-1} \right) + C$$

Exercise 7.5 → Int. by Partial Fraction.

Q.9 $I = \int \frac{3x+5}{\underbrace{x^3-x^2-x+1}} \cdot dx = \int \frac{3x+5}{x^2(x-1)-(x-1)} \cdot dx$

$$I = \int \frac{3x+5}{(\underbrace{x^2-1})(x-1)} \cdot dx = \int \frac{3x+5}{(\underbrace{x-1})^2(x+1)} \cdot dx$$

$$\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$\Rightarrow 3x+5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$$\Rightarrow 3x+5 = A(x^2-1) + B(x+1) + C(x^2-2x+1)$$

Comparison: $(x^2) \rightarrow 0 = A + C$ — (1)

$(x) \rightarrow 3 = B - 2C$ — (2)

Const. $\rightarrow 5 = -A + B + C$ — (3)

$$\boxed{A = -\frac{1}{2}} \quad \boxed{B = 4} \quad \boxed{C = \frac{1}{2}}$$

$$I = \int \frac{3x+5}{(x-1)^2(x+1)} \cdot dx$$

$$I = \int \left[\frac{-\frac{1}{2}}{(x-1)} + \frac{4}{(x-1)^2} + \frac{\frac{1}{2}}{(x+1)} \right] \cdot dx$$

$$I = -\left(\frac{1}{2}\right) \cdot \log|x-1| - \frac{4}{(x-1)} + \left(\frac{1}{2}\right) \log|x+1| + C$$

$$\boxed{I = \frac{1}{2} \log \left| \frac{x+1}{x-1} \right| - \frac{4}{(x-1)} + C}$$

Q.10 $I = \int \frac{2x-3}{(x^2-1)(2x+3)} \cdot dx = \int \frac{(2x-3)}{(x-1)(x+1)(2x+3)} \cdot dx$

$$\frac{2x-3}{(x-1)(x+1)(2x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3}$$

Proper Method

$$(2x-3) = A(2x^2+5x+3) + B(2x^2+x-3) + C(x^2-1)$$

Comparison.

$$\left. \begin{array}{l} (x^2) \rightarrow 0 = 2A + 2B + C \text{ --- (1)} \\ (x) \rightarrow 2 = 5A + B \text{ --- (2)} \\ (\text{Const.}) \rightarrow -3 = 3A - 3B - C \text{ --- (3)} \end{array} \right\} \begin{array}{l} A = -\frac{1}{10} \\ B = \frac{5}{2} \\ C = -\frac{24}{5} \end{array}$$

$$I = \int \frac{2x-3}{(x-1)(x+1)(2x+3)} \cdot dx$$

$$I = \int \left(\frac{-\frac{1}{10}}{x-1} + \frac{\frac{5}{2}}{x+1} + \frac{-\frac{24}{5}}{2x+3} \right) \cdot dx$$

$$I = -\frac{1}{10} \log|x-1| + \frac{5}{2} \log|x+1| - \frac{\frac{12}{5} \cdot \log|2x+3|}{2} + c$$

✓

$$\boxed{\text{Q.11}} \quad I = \int \frac{5x}{(x+1)(x^2-4)} \cdot dx$$

$$I = \int \frac{5x}{(x+1)(x-2)(x+2)} \cdot dx$$

$$I = \int \left(\frac{\left(\frac{5}{3}\right)}{x+1} + \frac{\left(\frac{5}{6}\right)}{x-2} - \frac{\left(\frac{5}{2}\right)}{x+2} \right) \cdot dx$$

$$\cancel{I} \quad I = \frac{5}{3} \log|x+1| + \frac{5}{6} \log|x-2| - \frac{5}{2} \log|x+2| + C$$

$$\boxed{\text{Q.12}} \quad I = \int \frac{x^3 + x + 1}{x^2 - 1} \cdot dx$$

$$\text{Deg. of } (N_r) = 3 > 2$$

$$\text{Deg. of } (D_r) = 2$$

Division

$$I = \int \left(x + \frac{2x+1}{x^2-1} \right) \cdot dx$$

$$\begin{array}{r} x^2-1 \overline{) x^3+x+1} \\ \underline{x^3-x} \\ 2x+1 \end{array}$$

$$I = \int \left[x + \frac{2x+1}{(x-1)(x+1)} \right] \cdot dx$$

$$\rightarrow \frac{A}{x-1} + \frac{B}{x+1}$$

$$I = \int \left[x + \frac{3/2}{(x-1)} + \frac{1/2}{x+1} \right] \cdot dx$$

$$I = \frac{x^2}{2} + \frac{3}{2} \log|x-1| + \frac{1}{2} \log|x+1| + C$$

Q.13

$$I = \int \frac{2}{(1-x)(1+x^2)} \cdot dx$$

$\rightarrow$ No Factors.

$$\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$$

$$\Rightarrow 2 = A(1+x^2) + (Bx+C)(1-x)$$

$$\Rightarrow 2 = A(1+x^2) + Bx - Bx^2 + C - Cx$$

Comparison:

$$\left. \begin{array}{l} (x^2) \rightarrow 0 = A - B \text{ --- (1)} \\ (x) \rightarrow 0 = B - C \text{ --- (2)} \\ \text{Const.} \rightarrow 2 = A + C \text{ --- (3)} \end{array} \right\} \begin{array}{l} A = 1 \\ B = 1 \\ C = 1 \end{array}$$

$$I = \int \frac{2}{(1-x)(1+x^2)} \cdot dx = \int \left[\frac{1}{1-x} + \frac{x+1}{1+x^2} \right] + C$$

$$I = \int \frac{1}{1-x} \cdot dx + \frac{1}{2} \int \frac{2x}{1+x^2} \cdot dx + \int \frac{1}{1+x^2} \cdot dx$$

$\xrightarrow{dt}$ $\xrightarrow{t}$

$$I = \frac{\log|1-x|}{(-1)} + \frac{1}{2} \log|1+x^2| + \tan^{-1}x + C$$

$$I = -\log|1-x| + \frac{1}{2} \log|1+x^2| + \tan^{-1}x + C$$

Q.14 $I = \int \frac{3x-1}{(x+2)^2} \cdot dx$

$$\frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

$$\Rightarrow (3x-1) = A(x+2) + B$$

Comparison, $(x) \rightarrow 3 = A$

Const $\rightarrow -1 = 2A + B$

$$\Rightarrow -1 = 6 + B \Rightarrow B = -7$$

$$I = \int \frac{3x-1}{(x+2)^2} \cdot dx = \int \left[\frac{3}{(x+2)} - \frac{7}{(x+2)^2} \right] \cdot dx$$

$$I = 3 \log|x+2| + 7 \left(\frac{1}{x+2} \right) + C$$

Q.15 $I = \int \frac{1}{x^4-1} \cdot dx = \int \frac{1}{(x^2-1)(x^2+1)} \cdot dx$

$$\left. \frac{1}{(x^2-1)(x^2+1)} \right\} \rightarrow (x^2=y) \rightarrow \left\{ \frac{1}{(y-1)(y+1)} = \frac{(\frac{1}{2})}{y-1} + \frac{(-\frac{1}{2})}{y+1} \right.$$

$$\boxed{\frac{1}{(x^2-1) \cdot (x^2+1)}} = \boxed{\frac{\frac{1}{2}}{x^2-1} - \frac{\frac{1}{2}}{x^2+1}}$$

$$I = \int \frac{1}{x^4-1} \cdot dx = \int \frac{1}{(x^2-1) \cdot (x^2+1)} \cdot dx = \int \left(\frac{\frac{1}{2}}{x^2-1} - \frac{\frac{1}{2}}{x^2+1} \right) \cdot dx$$

$$I = \underbrace{\frac{1}{2} \times \frac{1}{2}}_{\left(\frac{1}{4}\right)} \log \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \tan^{-1} x + C$$

Exercise 7.5

Int. by Partial Fraction

Q.16

$$I = \int \frac{1}{x(x^n+1)} \cdot dx$$

$$\frac{d(x^n)}{dx} = n \cdot x^{n-1}$$

$$I = \int \frac{x^{n-1}}{\underbrace{x(x^n+1)} \cdot \underbrace{x^{n-1}}} \cdot dx$$

$$I = \int \frac{x^{n-1}}{x^n \cdot (x^n+1)} \cdot dx$$

Substitution,

$$\text{let } x^n = t$$

$$\Rightarrow n \cdot x^{n-1} \cdot dx = dt$$

$$\Rightarrow \boxed{x^{n-1} \cdot dx = \frac{dt}{n}}$$

$$I = \frac{1}{n} \int \frac{dt}{t \cdot (t+1)}$$

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$$

$$\Rightarrow 1 = A(t+1) + B(t)$$

Comparison,

$$\text{Coefficient of } t \rightarrow 0 = A + B$$

$$\text{Constant} \rightarrow \boxed{1 = A}, \boxed{B = -1}$$

$$I = \frac{1}{n} \int \frac{dt}{t(t+1)} = \frac{1}{n} \int \left(\frac{1}{t} - \frac{1}{t+1} \right) \cdot dt$$

$$I = \frac{1}{n} \left[\log |t| - \log |t+1| \right] + C$$

$$I = \frac{1}{n} \left\{ \log |x^n| - \log |x^n+1| \right\} + C$$

$$\boxed{I = \frac{1}{n} \log \left| \frac{x^n}{x^n+1} \right| + C}$$



$$\boxed{Q.17} \quad I = \int \frac{\cos x}{(1 - \sin x)(2 - \sin x)} \cdot dx$$

$$\text{Let } \sin x = t \\ \Rightarrow \cos x \, dx = dt$$

$$I = \int \frac{dt}{(1-t)(2-t)}$$

$$\frac{1}{(1-t)(2-t)} = \frac{A}{1-t} + \frac{B}{2-t}$$

Shortcut

$$A = \frac{1}{1} = 1 \checkmark$$

$$B = \frac{1}{-1} = -1 \checkmark$$

$$I = \int \frac{dt}{(1-t)(2-t)} = \int \left(\frac{1}{(1-t)} - \frac{1}{(2-t)} \right) \cdot dt$$

$$I = \frac{\log |1-t|}{-1} - \frac{\log |2-t|}{-1} + C$$

$$\Rightarrow I = -\log |1 - \sin x| + \log |2 - \sin x| + C$$

$$I = \log \left| \frac{2 - \sin x}{1 - \sin x} \right| + C \checkmark$$

$$\boxed{Q.18} \quad I = \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} \cdot dx$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} \xrightarrow{x^2=y} \frac{(y+1)(y+2)}{(y+3)(y+4)} = \frac{y^2+3y+2}{y^2+7y+12}$$

long Division

$$\frac{y^2+3y+2}{y^2+7y+12} = 1 - \frac{(4y+10)}{y^2+7y+12}$$

$$\begin{array}{r} y^2+7y+12 \overline{) y^2+3y+2} \\ - y^2+7y+12 \\ \hline -4y-10 \\ = -(4y+10) \end{array}$$

$$= 1 - \frac{4y+10}{(y+3)(y+4)} \rightarrow \text{Partial Fraction}$$

$$= 1 - \left(\frac{-2}{y+3} + \frac{6}{y+4} \right)$$

$$= 1 + \frac{2}{y+3} - \frac{6}{y+4}$$

$$(y = x^2)$$

$$= 1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}$$

$$I = \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} \cdot dx = \int \left(1 + \frac{2}{x^2+3} - \frac{6}{x^2+4} \right) \cdot dx$$

$x^2 + (\sqrt{3})^2$

$$I = x + 2 \times \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 6 \times \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$$

$$I = x + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 3 \tan^{-1}\left(\frac{x}{2}\right) + c$$

Q.19 $I = \int \frac{2x}{(x^2+1)(x^2+3)} \cdot dx$

Let $x^2 = t$

$\Rightarrow 2x \cdot dx = \underline{dt}$

$I = \int \frac{dt}{(t+1) \cdot (t+3)}$

$I = \int \left(\frac{(\frac{1}{2})}{t+1} + \frac{(\frac{-1}{2})}{t+3} \right) \cdot dt$

$I = \frac{1}{2} \int \frac{1}{t+1} \cdot dt - \frac{1}{2} \int \frac{1}{t+3} \cdot dt$

$I = \frac{1}{2} \log |t+1| - \frac{1}{2} \log |t+3| + C$

$I = \frac{1}{2} \log \left| \frac{t+1}{t+3} \right| + C$

$I = \frac{1}{2} \log \left| \frac{x^2+1}{x^2+3} \right| + C$

Q.20 $I = \int \frac{1}{x(x^4-1)} \cdot dx$

$I = \int \frac{x^3 \cdot dx}{x^4(x^4-1)}$

$\left(\begin{array}{l} x^4 = t \\ 4x^3 \cdot dx = dt \\ x^3 \cdot dx = \frac{dt}{4} \end{array} \right)$

$I = \frac{1}{4} \int \frac{dt}{t(t-1)}$

$\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$

$1 = A(t-1) + B(t)$

Comparison $(t) \rightarrow 0 = A+B$
Const. $\rightarrow 1 = -A$

$\left\{ \begin{array}{l} A = -1 \\ B = 1 \end{array} \right.$

$$I = \frac{1}{4} \int \frac{dt}{t(t-1)} = \frac{1}{4} \int \left(\frac{-1}{t} + \frac{1}{t-1} \right) \cdot dt$$

$$I = \frac{1}{4} \left[-\log|t| + \log|t-1| \right] + C$$

$$I = \frac{1}{4} \log \left| \frac{t-1}{t} \right| + C \quad (t = x^4)$$

$$I = \frac{1}{4} \log \left| \frac{x^4-1}{x^4} \right| + C$$

[Q.2] $I = \int \frac{1}{e^x-1} \cdot dx$ Hint: $e^x = t$

$$\text{let } e^x = t \quad \checkmark$$

$$\Rightarrow e^x \cdot dx = dt$$

$$\Rightarrow dx = \frac{dt}{e^x}$$

$$\Rightarrow dx = \frac{dt}{t} \quad \checkmark$$

$$I = \int \frac{dt}{t(t-1)}$$

$$I = \int \left(\frac{-1}{t} + \frac{1}{t-1} \right) \cdot dt$$

$$I = -\log|t| + \log|t-1| + C$$

$$I = \log \left| \frac{t-1}{t} \right| + C$$

$$(e^x = t)$$

$$I = \log \left| \frac{e^x-1}{e^x} \right| + C$$

Q.22 $\int \frac{x \cdot dx}{(x-1)(x-2)}$ equals

(A) $\log \left| \frac{(x-1)^2}{(x-2)} \right| + C$ (B) $\log \left| \frac{(x-2)^2}{x-1} \right| + C$

(C) $\log \left| \left(\frac{x-1}{x-2} \right)^2 \right| + C$ (D) $\log |(x-1)(x-2)| + C$

$$I = \int \frac{x}{(x-1)(x-2)} dx = \int \left(\frac{-1}{x-1} + \frac{2}{x-2} \right) dx$$

$$I = -\log|x-1| + 2 \log|x-2| + C$$

$$I = -\log|x-1| + \log|(x-2)^2| + C$$

$$I = \log \left| \frac{(x-2)^2}{(x-1)} \right| + C$$

$\log(m^n)$

$\log m - \log n$
 $= \log \frac{m}{n}$

Q.23 $\int \frac{dx}{x(x^2+1)}$ equals -

$$I = \int \frac{dx}{x(x^2+1)} \cdot x \cdot \frac{x}{x}$$

$$I = \int \frac{x dx}{x^2(x^2+1)}$$

$$\begin{aligned} x^2 &= t \\ 2x dx &= dt \\ x \cdot dx &= \frac{dt}{2} \end{aligned}$$

$$I = \frac{1}{2} \int \frac{dt}{t(t+1)} = \frac{1}{2} \int \left(\frac{1}{t} + \frac{-1}{t+1} \right) dt$$

$$I = \frac{1}{2} \left[\log|t| - \log|t+1| \right] + C = \frac{1}{2} \log \left| \frac{x^2}{x^2+1} \right| + C$$

$$I = \frac{1}{2} \left\{ \log |t| - \log |t+1| \right\} + C$$

$$t = x^2$$

$$I = \frac{1}{2} \log(x^2) - \frac{1}{2} \log(x^2+1) + C$$

$$I = \log|x| - \frac{1}{2} \log|x^2+1| + C$$

$$\log m^n = n \cdot \log m$$

Option - A

Integration by Parts / खण्डन: समकलन

$$u \rightarrow \text{I} \leftarrow f(x)$$

$$v \rightarrow \text{II} \leftarrow g(x)$$

$$\boxed{\int u \cdot v \cdot dx = \underbrace{u}_{\text{I}} \cdot \underbrace{\int v dx}_{\text{II}} - \int \left[\underbrace{\frac{du}{dx}}_{\text{I}} \cdot \underbrace{\int v dx}_{\text{II}} \right] \cdot dx}$$

$$\int \underbrace{f(x)}_{\text{I}} \cdot \underbrace{g(x)}_{\text{II}} \cdot dx = f(x) \cdot \int g(x) \cdot dx - \int \left(\underbrace{f'(x)}_{\text{I}} \cdot \underbrace{\int g(x) \cdot dx}_{\text{II}} \right) \cdot dx$$
$$\int \text{I} \cdot \text{II} \cdot dx = \underbrace{(\text{I}) \cdot (\text{Int. of II})}_{\text{I} \cdot \text{II}} - \text{Int. of } \left(\underbrace{\text{Diff. of I}}_{\text{I}} \times \underbrace{\text{Int. of II}}_{\text{II}} \right)$$

How to Decide I & II?

ILATE

Inverse trigo. Fn. ($\sin^{-1}x, \cos^{-1}x \dots$)

Logarithmic ($\log x$)

Algebraic ($x, x^2, x^3, \frac{1}{x}, \sqrt{x}, \underline{1, 2, \dots}$)

Trigo. ($\sin x, \cos x \dots$)

Exponential ($e^x, a^x, 2^x$)

↑ I

Diff = easy

Int. = tough

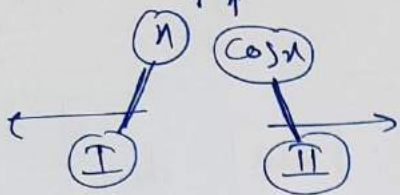
↓ II

Diff. = tough

int. = easy

e.g. $I = \int \underbrace{x}_{\textcircled{A}} \cdot \underbrace{\cos x}_{\textcircled{B}} dx$

ILATE



$$I = \int \underbrace{x}_{\text{I}} \underbrace{\cos x}_{\text{II}} \cdot dx$$

$$I = x \int \cos x \cdot dx$$

$$- \int \left(\frac{d(x)}{dx} \cdot \int \cos x \cdot dx \right) \cdot dx$$

$$I = x(\sin x) - \int (1 \cdot \sin x) \cdot dx$$

Correct

$$\boxed{I = x \sin x + \cos x + c}$$

wrong.

$$I = \int \underbrace{x}_{\text{II}} \underbrace{\cos x}_{\text{I}} \cdot dx$$

$$I = \cos x \cdot \int x \cdot dx - \int (-\sin x) \cdot \frac{x^2}{2} \cdot dx$$

e.g. $I = \int \log x \cdot dx$

$$I = \int \underbrace{\log x}_{\text{I}} \cdot \underbrace{1}_{\text{II}} \cdot dx$$

$$I = \log x \cdot \int 1 \cdot dx - \int \left[\frac{d(\log x)}{dx} \cdot \int 1 \cdot dx \right] \cdot dx$$

$$I = \log x \cdot (x) - \int \left(\frac{1}{x} \cdot x \right) \cdot dx$$

$$I = \underline{x \log x - x} + c$$

$$\boxed{\int \log x \cdot dx = x(\log x - 1) + c}$$

e.g.

$$I = \int \sin^{-1} x \cdot dx$$

आपके लिए

only try

e.g.

$$I = \int e^x \sin x \cdot dx$$

$e^x \rightarrow I$
 $\sin x \rightarrow II$

I
L
A

$T \rightarrow I$
 $E \rightarrow II$

$\uparrow I$

$L II$

$$I = \int e^x \sin x \cdot dx$$

$$I = \sin x \int e^x \cdot dx - \int \left[\frac{d(\sin x)}{dx} \cdot \int e^x \cdot dx \right] \cdot dx$$

$$I = \sin x \cdot e^x - \int (\cos x \cdot e^x) \cdot dx$$

$$I = \sin x \cdot e^x - \cos x \cdot e^x + \int (-\sin x) \cdot e^x \cdot dx$$

$$I = e^x \cdot \sin x - e^x \cos x - \int \sin x \cdot e^x \cdot dx$$

Question

$$I = e^x \sin x - e^x \cos x - I$$

$$2I = e^x (\sin x - \cos x)$$

$$I = \frac{e^x}{2} (\sin x - \cos x) + C$$

$\int u \cdot v$

Note

Remember

$$\int e^x (f(x) + f'(x)) \cdot dx = e^x \cdot f(x) + C$$

Proof $I = \int e^x (f(x) + f'(x)) \cdot dx$

$$I = \int e^x \cdot f(x) \cdot dx + \int e^x \cdot f'(x) \cdot dx$$

ILATE $\textcircled{E}$ II $\textcircled{I}$ By parts

$$I = f(x) \cdot e^x - \int (f'(x) \cdot e^x) \cdot dx + \int e^x \cdot f'(x) \cdot dx$$

$$I = e^x \cdot f(x) + C$$

e.g. $I = \int e^x \left(\tan^{-1} x + \frac{1}{1+x^2} \right) \cdot dx = e^x \tan^{-1} x + C$

$$\int e^x (f(x) + f'(x)) \cdot dx = e^x \cdot f(x) + C$$

e.g. $I = \int \frac{(x^2+1)}{(x+1)^2} \cdot e^x \cdot dx = \int \left(\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2} \right) \cdot e^x \cdot dx$

$$I = \int \left[\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2} \right] \cdot e^x \cdot dx$$

$$I = \int \left[\frac{(x-1)(x+1)}{(x+1)^2} + \frac{2}{(x+1)^2} \right] \cdot e^x \cdot dx \Rightarrow I = \int \left(\frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right) \cdot e^x \cdot dx$$

$$I = e^x \cdot f(x) + C \quad \checkmark$$

$$I = e^x \cdot \left(\frac{x-1}{x+1} \right) + C \quad \checkmark$$

Exercise 7.6 → Integration by Parts.

$$\int \underset{\text{I}}{u} \cdot \underset{\text{II}}{v} \cdot dx$$

$$\int \text{I} \cdot \text{II} \cdot dx = \text{I} \int \text{II} \cdot dx - \int \left[\frac{d(\text{I})}{dx} \cdot \int \text{II} \cdot dx \right] \cdot dx$$

I & II

Inverse Trigo. Func.

Log.

Algebraic

Trigo.

Expo.

↑ (I)

Diff. easy

Int. tough

↓ (II)

Diff. tough

Int. easy

Exercise 7.6

Q.1 $I = \int \underset{\text{I}}{x} \cdot \underset{\text{II}}{\sin x} \cdot dx$

ILATE
(I) (II)

$$\int \text{I} \cdot \text{II} = \text{I} \int \text{II} - \int (\text{I}' \int \text{II})$$

$$I = x \int \sin x \cdot dx - \int \left[\frac{d(x)}{dx} \cdot \int \sin x \cdot dx \right] \cdot dx$$

$$I = x(-\cos x) - \int 1 \cdot (-\cos x) \cdot dx$$

$$I = -x \cos x + (\sin x) + C$$

Q.2 $I = \int \overset{I}{x} \cdot \overset{II}{\sin 3x} \cdot dx$

$\swarrow \quad \searrow$
 I L A T E
 (I) (II)

$$\boxed{\int I \cdot II = I \int II - \int [I' \cdot \int II]}$$

$$I = x \int \sin 3x \cdot dx - \int \left[\underset{\rightarrow 1}{\frac{d(x)}{dx}} \cdot \int \sin 3x \cdot dx \right] \cdot dx$$

$$I = x \left(\frac{-\cos 3x}{3} \right) - \int \left(\frac{-\cos 3x}{3} \right) \cdot dx$$

$$I = -\frac{x}{3} \cos 3x + \frac{1}{3} \int \cos 3x \cdot dx$$

$$\boxed{I = -\frac{x}{3} \cos 3x + \frac{1}{3} \left(\frac{\sin 3x}{3} \right) + C}$$

Q.3 $I = \int \overset{I}{x^2} \cdot \overset{II}{e^x} \cdot dx$

$\swarrow \quad \searrow$
 I L A T E
 (I) (II)

$$\boxed{\int I \cdot II = I \int II - \int [I' \cdot \int II]}$$

by int. by parts.

$$\boxed{\int e^x \cdot dx = e^x + C}$$

$$I = x^2 \cdot \int e^x \cdot dx - \int \underbrace{\left(\overset{I}{2x} \right) \cdot \left(\overset{II}{e^x} \right)}_{\text{I L A T E}} \cdot dx$$

$$I = x^2 e^x - \left\{ 2x \cdot \int e^x \cdot dx - \int (2 \cdot e^x) \cdot dx \right\} \quad \text{(again by Parts)}$$

$$I = x^2 e^x - \left[2x e^x - 2 e^x \right] + C$$

$$I = x^2 e^x - 2x e^x + 2 e^x + C$$

$$\boxed{I = e^x (x^2 - 2x + 2) + C}$$

Q.4 $I = \int \frac{I}{x} \cdot \log x \cdot dx$

ILATE

$$\int \log x \cdot dx = ?$$

$$\boxed{\int I \cdot II = I \int II - \int (I' \cdot II)} \quad \underline{\underline{\int (I' \cdot II)}}$$

~~$I =$~~ $I = \log x \int x \cdot dx - \int \left\{ \frac{\frac{d}{dx}(\log x)}{\frac{d}{dx}} \cdot \int x \cdot dx \right\} \cdot dx$

$$\Rightarrow I = \log x \cdot \frac{x^2}{2} - \int \left(\frac{1}{x} \right) \cdot \left(\frac{x^2}{2} \right) \cdot dx \quad = \int \frac{x}{2} \cdot dx$$

$$\Rightarrow I = \frac{x^2}{2} \log x - \frac{1}{2} \left(\frac{x^2}{2} \right) + C$$

$$\boxed{I = \frac{x^2}{2} \log x - \frac{x^2}{4} + C}$$

Q.5 $I = \int \frac{I}{x} \cdot \log 2x \cdot dx$

ILATE

by integration by parts

$$I = \log 2x \int x \cdot dx - \int \left[\frac{\frac{d}{dx}(\log 2x)}{\frac{d}{dx}} \cdot \int x \cdot dx \right] \cdot dx$$

$$I = \log 2x \cdot \frac{x^2}{2} - \int \left(\frac{1}{2x} \right) \cdot \frac{x^2}{2} \cdot dx$$

$$\boxed{I = \frac{x^2}{2} \cdot \log 2x - \frac{1}{2} \left(\frac{x^2}{2} \right) + C}$$

Q.6

$$I = \int x^2 \cdot \log x \cdot dx$$

ILATE
②

$$\int I \cdot II = I \int II$$

$$- \int \{ I' \cdot \int II \}$$

$$I = \log x \int x^2 \cdot dx - \int \left\{ \frac{d(\log x)}{dx} \cdot \int x^2 \cdot dx \right\} \cdot dx$$

$$I = \log x \cdot \frac{x^3}{3} - \int \left(\frac{1}{x} \cdot \frac{x^3}{3} \cdot x^2 \right) \cdot dx$$

$$I = \log x \cdot \frac{x^3}{3} - \frac{1}{3} \left(\frac{x^3}{3} \right) + C$$

$$I = \frac{x^3}{3} \cdot \log x - \frac{x^3}{9} + C$$

Exercise-7.6 → Int. by Parts.

Q7 to Q15

$$\int \underbrace{u \cdot v}_{I \cdot II} \cdot du$$

$$\int I \cdot II \cdot du = I \int II \cdot du - \int \left(\frac{d(I)}{du} \cdot \int II \cdot du \right) \cdot du$$

I → Inverse trigo.

L → log

A → Algebraic ($x, x^2, \dots, 1, 2, 3$)

T → Trigo.

E → Expo.

↑ I Diff. easy
Int. tough

↓ II Diff. tough
Int. easy

Exercise 7.6 class 12

Q. 7 $I = \int \overset{\textcircled{II}}{x} \cdot \overset{\textcircled{I}}{\sin^{-1}x} \cdot du$
 $\swarrow \searrow$
 ILATE

$$\int \underbrace{I \cdot II} = \underbrace{I \int II} - \int \left(\underbrace{I'} \cdot \underbrace{\int II} \right)$$

Int. by Parts

$$I = \sin^{-1}x \cdot \underline{\int x \, du} - \int \left(\frac{d(\sin^{-1}x)}{du} \cdot \int x \, du \right) \cdot du$$

$$I = \sin^{-1}x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} \cdot du$$

$$I = \sin^{-1}x \cdot \frac{x^2}{2} + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} \cdot du$$

$$I = \underline{\underline{\frac{x^2}{2} \cdot \sin^{-1}x}} + \frac{1}{2} \int \frac{1-x^2}{\sqrt{1-x^2}} \cdot du - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \cdot du$$

$\swarrow$
 $\sin^{-1}x$

$$I = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{(1-x^2)}{\sqrt{1-x^2}} \cdot dx - \frac{1}{2} (\sin^{-1} x)$$

$$I = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1-x^2} \cdot dx$$

$\checkmark$ $\downarrow$ $\downarrow$ $\downarrow$
 $x = \sin \theta$
Substitution
 $\int \sqrt{a^2 - x^2}$ $(a=1)$
 $\checkmark$

Remember

$$\int \sqrt{a^2 - x^2} \cdot dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

$$\Rightarrow I = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right] + C$$

$$I = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + \frac{1}{4} \sin^{-1} x + C$$

$$I = \sin^{-1} x \cdot \left(\frac{x^2}{2} - \frac{1}{2} + \frac{1}{4} \right) + \frac{x \sqrt{1-x^2}}{4} + C$$

$$I = \sin^{-1} x \cdot \left(\frac{2x^2 - 1}{4} \right) + \frac{x \sqrt{1-x^2}}{4} + C$$

Q.8 $I = \int \overset{\textcircled{II}}{x} \cdot \overset{\textcircled{I}}{\tan^{-1}x} \cdot dx$
 $\swarrow \searrow$
 ILATE

Int. by parts.

$$\boxed{\begin{aligned} \int I \cdot II &= I \cdot \int II \\ &- \int (I' \cdot \int II) \end{aligned}}$$

$$I = \tan^{-1}x \cdot \int x \cdot dx - \int \left(\frac{d(\tan^{-1}x)}{dx} \cdot \int x dx \right) \cdot dx$$

$$I = \tan^{-1}x \cdot \frac{x^2}{2} - \int \left(\frac{1}{1+x^2} \cdot \frac{x^2}{2} \right) \cdot dx$$

$$I = \tan^{-1}x \cdot \frac{x^2}{2} - \frac{1}{2} \int \frac{\underline{1+x^2} - 1}{1+x^2} \cdot dx$$

$$I = \frac{x^2}{2} \tan^{-1}x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) \cdot dx$$

$$I = \frac{x^2}{2} \cdot \tan^{-1}x - \frac{1}{2} \left(x - \tan^{-1}x \right) + C$$

$$\boxed{I = \frac{x^2}{2} \tan^{-1}x - \frac{x}{2} + \frac{1}{2} \tan^{-1}x + C}$$

Q.9 $I = \int \overset{\textcircled{II}}{x} \cdot \overset{\textcircled{I}}{\cos^{-1}x} \cdot dx$
 $\swarrow \searrow$
 ILATE

Similar to Q.7

$$I = \cos^{-1}x \cdot \int x \cdot dx - \int \left(\frac{d(\cos^{-1}x)}{dx} \cdot \int x dx \right) \cdot dx$$

$$I = \cos^{-1}x \cdot \frac{x^2}{2} - \int \frac{(-1)}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} \cdot dx$$

$$I = \cos^{-1}x \cdot \frac{x^2}{2} - \frac{1}{2} \int \frac{\underline{1-x^2} - 1}{\sqrt{1-x^2}} \cdot dx$$

$$I = \frac{x^2}{2} \cos^{-1} x - \frac{1}{2} \int \left(\frac{(1-x^2)^{1/2}}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right) \cdot dx$$

$$I = \frac{x^2}{2} \cos^{-1} x - \frac{1}{2} \int \left(\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right) \cdot dx$$

$$\int \sqrt{a^2 - x^2} \cdot dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x$$

$$\int \frac{1}{\sqrt{1-x^2}} \cdot dx = \sin^{-1} x + C$$

$$I = \frac{x^2}{2} \cos^{-1} x - \frac{1}{2} \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x - \sin^{-1} x \right] + C$$

$$I = \frac{x^2}{2} \cos^{-1} x - \frac{x}{4} \sqrt{1-x^2} + \frac{1}{4} \sin^{-1} x + C$$

$$\frac{\pi}{2} - \cos^{-1} x$$

$$I = \frac{x^2}{2} \cos^{-1} x - \frac{x}{4} \sqrt{1-x^2} + \frac{1}{4} \left(\frac{\pi}{2} - \cos^{-1} x \right) + C$$

$$I = \frac{x^2}{2} \cos^{-1} x - \frac{x}{4} \sqrt{1-x^2} + \left(\frac{\pi}{8} \right) - \frac{\cos^{-1} x}{4} + C$$

$$I = \cos^{-1} x \left(\frac{x^2}{2} - \frac{1}{4} \right) - \frac{x}{4} \sqrt{1-x^2} + C$$

$$\text{Constant } = C'$$

$$I = \cos^{-1} x \cdot \left(\frac{2x^2 - 1}{4} \right) - \frac{x \sqrt{1-x^2}}{4} + C'$$

Q.10 $I = \int (\sin^{-1} x)^2 \cdot dx$

$\int \log u \cdot du \cdot \frac{1}{u}$
 $\frac{I}{II}$

$I = \int \overset{II}{1} \cdot \overset{I}{(\sin^{-1} x)^2} \cdot dx$
 ILATE

$\int I \cdot II = I \cdot \int II - \int (I' \cdot \int II)$

Int by parts

$I = (\sin^{-1} x)^2 \cdot \int 1 \cdot dx - \int \left(\frac{d(\sin^{-1} x)^2}{dx} \cdot \int 1 \cdot dx \right) \cdot dx$

$I = (\sin^{-1} x)^2 \cdot x - \int \left(2 \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} \cdot x \right) \cdot dx$

$I = x(\sin^{-1} x)^2 - \int \sin^{-1} x \cdot \frac{2x}{\sqrt{1-x^2}} \cdot dx$

$\frac{\sin^{-1} x \cdot \frac{2x}{\sqrt{1-x^2}}}{I \quad II}$
 ILATE
 $\sin^{-1} x = t$ (Put)

$\int \frac{2x}{\sqrt{1-x^2}} \cdot dx$
 $= \int \frac{-dt}{\sqrt{t}}$
 $= - \int t^{-1/2} \cdot dt = - \frac{t^{1/2}}{1/2} = -2\sqrt{1-x^2}$
 $1-x^2 = t$
 $-2x dx = dt$
 $2x dx = -dt$

$I = x(\sin^{-1} x)^2 - \int \sin^{-1} x \cdot \frac{2x}{\sqrt{1-x^2}} \cdot dx$

$I = x(\sin^{-1} x)^2 - \left[\sin^{-1} x \int \frac{2x}{\sqrt{1-x^2}} \cdot dx - \int \left(\frac{d(\sin^{-1} x)}{dx} \cdot \int \frac{2x}{\sqrt{1-x^2}} \cdot dx \right) \cdot dx \right]$

$$I = x(\sin^{-1}x)^2 - \sin^{-1}x \cdot \underline{(-2\sqrt{1-x^2})} + \int \frac{1}{\cancel{\sqrt{1-x^2}}} \cdot \underline{(-2\sqrt{1-x^2})} \cdot dx$$

$$I = x(\sin^{-1}x)^2 + 2\sqrt{1-x^2} \cdot \sin^{-1}x - 2\int 1 \cdot dx$$

$$I = x(\sin^{-1}x)^2 + 2\sqrt{1-x^2} \cdot \sin^{-1}x - 2x + C$$

Q.11

$$I = \int \frac{\overset{\text{II}}{x} \cdot \overset{\text{I}}{\cos^{-1}x}}{\sqrt{1-x^2}} \cdot dx$$

I L A T E

$$\int I \cdot II = I \int II - \int (I' \cdot II)$$

$$I = \cos^{-1}x \cdot \int \frac{x}{\sqrt{1-x^2}} \cdot dx - \int \left(\frac{d(\cos^{-1}x)}{dx} \cdot \int \frac{x}{\sqrt{1-x^2}} \cdot dx \right) \cdot dx$$

Here

$$\int \frac{x}{\sqrt{1-x^2}} \cdot dx = -\frac{1}{2} \int \frac{dt}{\sqrt{t}} = -\frac{1}{2} \int t^{-\frac{1}{2}} \cdot dt = -\frac{1}{2} \left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}} \right) = -t^{\frac{1}{2}}$$

$$1-x^2=t$$

$$-2x dx = dt$$

$$\textcircled{x dx} = \frac{dt}{2}$$

$$= \underline{-\sqrt{1-x^2}}$$

$$\frac{d(\cos^{-1}x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow I = \underline{\cos^{-1}x} \cdot \underline{(-\sqrt{1-x^2})} - \int \left(\frac{1}{\sqrt{1-x^2}} \right) \cdot \left(\cancel{+ \sqrt{1-x^2}} \right) \cdot dx$$

$$I = -\sqrt{1-x^2} \cdot \cos^{-1}x - \int 1 \cdot dx$$

$$I = -\sqrt{1-x^2} \cdot \cos^{-1}x - x + C$$

[Q.12]

$$I = \int \underbrace{x}_{\text{I}} \cdot \underbrace{\sec^2 x}_{\text{II}} \cdot \underbrace{dx}_{\text{E}} \quad \text{ILATE}$$

Int. by parts.

$$I = x \int \sec^2 x \cdot dx - \int \left(\frac{d(x)}{dx} \cdot \int \sec^2 x \cdot dx \right) \cdot dx$$

$$I = x \cdot \tan x - \int (\tan x) \cdot dx$$

$$I = x \cdot \tan x - (-\log |\cos x|) + c$$

$$\boxed{I = x \tan x + \log |\cos x| + c}$$

[Q.13]

$$I = \int \underbrace{1}_{\text{I}} \cdot \underbrace{\tan^{-1} x}_{\text{II}} \cdot \underbrace{dx}_{\text{E}} \quad \text{ILATE}$$

Int. by parts. $I = \tan^{-1} x \cdot \int 1 dx - \int \left(\frac{d(\tan^{-1} x)}{dx} \cdot \int 1 dx \right) \cdot dx$

$$\Rightarrow I = \tan^{-1} x \cdot (x) - \int \frac{1}{1+x^2} \cdot x \cdot dx$$

$$(1+x^2=t) \Rightarrow 2x dx = dt$$

$$\Rightarrow I = x \cdot \tan^{-1} x - \frac{1}{2} \int \frac{dt}{t}$$

$$\frac{x dx}{2} = \frac{dt}{2}$$

$$\Rightarrow I = x \tan^{-1} x - \frac{1}{2} \log |t| + c$$

$$\Rightarrow \boxed{I = x \tan^{-1} x - \frac{1}{2} \log |1+x^2| + c}$$

Q.14 $I = \int \frac{\text{II}}{\text{I}} \cdot \frac{\text{I}}{(\log x)^2} \cdot dx$

I L A T E

Int. by parts

$$\boxed{\begin{aligned} \int \text{I} \cdot \text{II} &= \text{I} \int \text{II} \\ &- \int (\text{I}' \cdot \int \text{II}) \end{aligned}}$$

$$I = (\log x)^2 \cdot \int x dx - \int \left(\frac{d(\log x)^2}{dx} \cdot \int x dx \right) \cdot dx$$

$$I = (\log x)^2 \cdot \frac{x^2}{2} - \int \cancel{2}(\log x) \cdot \cancel{\frac{1}{x}} \cdot \frac{x^2}{\cancel{2}} \cdot dx$$

$$I = (\log x)^2 \cdot \frac{x^2}{2} - \int \underbrace{\log x}_{\text{I}} \cdot \underbrace{\frac{x^2}{2}}_{\text{II}} \cdot dx$$

I L A T E

Again int. by parts

$$I = \frac{x^2}{2} \cdot (\log x)^2 - \left[\log x \cdot \int x^{\frac{1}{2}} dx - \int \left(\frac{d(\log x)}{dx} \cdot \int x dx \right) \cdot dx \right]$$

$$I = \frac{x^2}{2} (\log x)^2 - \left[\log x \cdot \frac{x^2}{2} - \int \left(\frac{1}{x} \cdot \frac{x^2}{2} \right) \cdot dx \right]$$

$$I = \frac{x^2}{2} (\log x)^2 - \log x \cdot \frac{x^2}{2} + \frac{1}{2} \int x dx$$

$$\boxed{I = \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \cdot \log x + \frac{1}{2} \left(\frac{x^2}{2} \right) + C}$$

Q.15

$$I = \int \frac{x^{\frac{\pi}{2}}}{(x^2+1)} \log x \cdot dx$$

ILATE

$$\boxed{\begin{aligned} &\int I \cdot II \\ &= I \int II - \int (I' \cdot \int II) \cdot dx \end{aligned}}$$

Int. by parts.

$$I = \log x \int (x^2+1) \cdot dx - \int \left(\frac{d(\log x)}{dx} \cdot \int (x^2+1) dx \right) \cdot dx$$

$$I = \log x \cdot \left(\frac{x^3}{3} + x \right) - \int \frac{1}{x} \cdot \left(\frac{x^3}{3} + x \right) \cdot dx$$

$$I = \left(\frac{x^3}{3} + x \right) \log x - \int \left(\frac{x^2}{3} + 1 \right) \cdot dx$$

$$I = \left(\frac{x^3}{3} + x \right) \cdot \log x - \frac{1}{3} \left(\frac{x^3}{3} \right) - (x) + C$$

$$\boxed{I = \left(\frac{x^3}{3} + x \right) \cdot \log x - \frac{x^3}{9} - x + C}$$

Exercise 7.6

Q16 — Q24

$$\int e^x (f(x) + f'(x)) \cdot dx = e^x \cdot f(x) + C$$

Q.16

$$I = \int e^x (\sin x + \cos x) \cdot dx = \int e^x (f(x) + f'(x)) \cdot dx$$

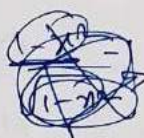
let $f(x) = \sin x$
 $f'(x) = \cos x$

$$= e^x f(x) + C$$

$$= e^x \cdot \sin x + C$$

Q.17

$$I = \int \frac{x e^x}{(1+x)^2} \cdot dx$$



$$I = \int e^x \cdot \frac{(1+x-1)}{(1+x)^2} \cdot dx$$

$$I = \int e^x \left(\frac{(1+x)^1}{(1+x)^2} - \frac{1}{(1+x)^2} \right) \cdot dx$$

$$I = \int e^x \left(\frac{1}{1+x} - \frac{1}{(1+x)^2} \right) \cdot dx$$

$$\int e^x (f(x) + f'(x)) \cdot dx = e^x \cdot f(x) + C$$

let $f(x) = \frac{1}{1+x}$
 $f'(x) = -\frac{1}{(1+x)^2}$

$$I = e^x \cdot \frac{1}{1+x} + C$$

$$I = \frac{e^x}{1+x} + C$$

(Q.18)

$$I = \int e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) \cdot dx$$

$$\sin x = 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}$$

$$1 + \cos x = 1 + 2 \cos^2 \frac{x}{2}$$

$$\int e^x (f(x) + f'(x))$$

$$I = \int e^x \left(\frac{1 + 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) \cdot dx$$

$$I = \int e^x \left(\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) \cdot dx$$

$$\int e^x (f(x) + f'(x)) \cdot dx = e^x \cdot f(x) + c$$

let $f(x) = \tan \left(\frac{x}{2} \right)$

$$f'(x) = \sec^2 \frac{x}{2} \cdot \frac{1}{2} =$$

$$I = e^x \cdot f(x) + c = e^x \cdot \tan \frac{x}{2} + c$$

(Q.19)

$$I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) \cdot dx$$

$$\Rightarrow \int e^x (f(x) + f'(x)) \cdot dx = e^x \cdot f(x) + c$$

$$I = e^x \cdot \frac{1}{x} + c$$

$$\boxed{\text{Q.20}} \quad I = \int \frac{(x-3)e^x}{(x-1)^3} \cdot dx$$

$$I = \int \frac{(x-1-2)}{(x-1)^3} \cdot e^x \cdot dx$$

$$I = \int e^x \left(\frac{(x-1)'}{(x-1)^3} - \frac{2}{(x-1)^3} \right) \cdot dx$$

$$I = \int e^x \left(\frac{1}{(x-1)^2} + \frac{-2}{(x-1)^3} \right) \cdot dx$$

$$\boxed{\int e^x (f(x) + f'(x)) \cdot dx = e^x f(x) + C}$$

$$\int e^x (f(x) + f'(x))$$

↓

$$e^x \cdot f(x) + C$$

$$\begin{aligned} f(x) &= \frac{1}{(x-1)^2} \\ &= (x-1)^{-2} \\ f'(x) &= (-2) \cdot (x-1)^{-3} \\ &= \frac{-2}{(x-1)^3} \end{aligned}$$

$$I = e^x \cdot \left(\frac{1}{(x-1)^2} \right) + C$$

$$\boxed{\text{Q.21}} \quad I = \int e^{2x} \cdot \sin x \cdot dx$$

ILATE

Int. by parts.

$$\boxed{\int I \cdot II = I \int II - \int (I' \cdot \int II) \cdot dx}$$

$$\Rightarrow I = \sin x \int e^{2x} \cdot dx - \int \left(\frac{d(\sin x)}{dx} \cdot \int e^{2x} \cdot dx \right) \cdot dx$$

$$\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \frac{1}{2} \int \cos x \cdot e^{2x} \cdot dx$$

ILATE

$$I = \sin x \cdot \frac{e^{2x}}{2} - \frac{1}{2} \int \cos x \cdot e^{2x} \cdot dx$$

I **II**

$$I = \sin x \cdot \frac{e^{2x}}{2} - \frac{1}{2} \left[\cos x \cdot \underbrace{\int e^{2x} \cdot dx}_{\frac{e^{2x}}{2}} - \int \left(\frac{d(\cos x)}{dx} \right) \cdot \frac{e^{2x}}{2} \cdot dx \right]$$

$$I = \sin x \cdot \left(\frac{e^{2x}}{2} \right) - \frac{1}{2} \cos x \cdot \left(\frac{e^{2x}}{2} \right) - \frac{1}{4} \int \sin x \cdot e^{2x} \cdot dx$$

$\downarrow$
'I' = Question

$$I = \frac{e^{2x}}{2} \left(\sin x - \frac{1}{2} \cos x \right) - \frac{1}{4} I$$

$$\Rightarrow I + \frac{I}{4} = \frac{e^{2x}}{2} \left(\frac{\sin x}{1} - \frac{\cos x}{2} \right)$$

$$\Rightarrow \frac{5I}{4} = \frac{e^{2x}}{2} \left(\frac{2\sin x - \cos x}{2} \right)$$

$$\Rightarrow I = \frac{e^{2x}}{5} (2\sin x - \cos x) + C$$

Q.22 $I = \int \sin^{-1} \left(\frac{2x}{1+x^2} \right) \cdot dx$

$$\begin{aligned} \sin 2\theta &= 2 \sin \theta \cos \theta \\ \sin 2\theta &= \frac{2 \tan \theta}{1 + \tan^2 \theta} \end{aligned}$$

$\theta = \tan^{-1}(x)$

Substitution: $x = \tan \theta$

$dx = \sec^2 \theta \cdot d\theta$

$\cancel{\sin^{-1}}(\cancel{\tan \theta})$

$\cancel{\sin^{-1}}(\cancel{\sin \theta})$

$\begin{aligned} 1+x^2 &\rightarrow x = \tan \theta \\ 1-x^2 &\rightarrow x = \sin \theta \end{aligned}$

$$I = \int \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) \cdot \sec^2 \theta \cdot d\theta$$

$$I = \int \cancel{\sin^{-1}} \left(\cancel{\sin 2\theta} \right) \cdot \sec^2 \theta \cdot d\theta$$

$$I = \int \overset{I}{2\theta} \cdot \overset{II}{\sec^2 \theta} \cdot d\theta$$

I L A T E

$$\begin{aligned} \int I \cdot II &= I \cdot \int II \\ &- \int (I' \cdot \int II) \cdot d\theta \end{aligned}$$

$$I = 2\theta \cdot \int \sec^2 \theta \cdot d\theta - \int \left(\frac{d(2\theta)}{d\theta} \cdot \int \sec^2 \theta \cdot d\theta \right) \cdot d\theta$$

$$I = 2\theta \cdot \tan \theta - \int 2 \cdot \tan \theta \cdot d\theta$$

$\log$

$$I = 2\theta \tan \theta - 2 \cdot \log |\sec \theta| + C$$

$$I = 2\theta \tan \theta - \log |\sec^2 \theta| + C = 2\theta \cdot \tan \theta - \log \left| \frac{1+x^2}{1-x^2} \right| + C$$

$$I = 2 \tan^{-1} x \cdot (x) - \log |1+x^2| + C$$

Q.23 $I = \int x^2 \cdot e^{x^3} \cdot dx$

Substitution

Let $x^3 = t$
 $\Rightarrow 3x^2 \cdot dx = dt$
 $\Rightarrow \underline{x^2 dx} = \frac{dt}{3}$

$I = \frac{1}{3} \int e^t \cdot dt$
 $I = \frac{1}{3} (e^t) + C$

$I = \frac{e^{x^3}}{3} + C$ option A

Q.24 $I = \int e^x \cdot \sec x (1 + \tan x) \cdot dx$ equals —

$\int e^x \left(\frac{f(x)}{f'(x)} + f'(x) \right) \cdot dx = e^x \cdot \underline{f(x)} + C$

$I = \int e^x \left(\underline{\sec x} + \underline{\sec x \cdot \tan x} \right) \cdot dx$

$I = e^x \cdot \sec x + C$ ✓

option - B

Integrals of Some more Types

$\sqrt{Q}$
 $\frac{1}{21}$ Quadratic

$$\int \sqrt{x^2 - a^2} \cdot dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} \cdot dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\int \sqrt{a^2 - x^2} \cdot dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

Proof: $\int \sqrt{x^2 - a^2} \cdot dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$

$$I = \int \underbrace{1}_{II} \cdot \underbrace{\sqrt{x^2 - a^2}}_I \cdot dx$$

ILATE

(I) $\rightarrow$ Diff. easy, int. tough.

(II) $\rightarrow$ Diff. tough, int. easy

$$\int I \cdot II = \underline{I \cdot \int II} - \int \left(\underline{I' \cdot \int II} \right) \cdot dx$$

$$I = \sqrt{x^2 - a^2} \cdot \int 1 \cdot dx - \int \left(\frac{d(\sqrt{x^2 - a^2})}{dx} \cdot \int 1 \cdot dx \right) \cdot dx$$

$$I = \sqrt{x^2 - a^2} \cdot x - \int \frac{x}{\sqrt{x^2 - a^2}} \cdot x \cdot dx$$

$$I = x \sqrt{x^2 - a^2} - \int \frac{a^2 + \cancel{x^2} - a^2}{\sqrt{\cancel{x^2} - a^2}} \cdot dx$$

$$I = x \sqrt{x^2 - a^2} - \int \frac{a^2 \cdot dx}{\sqrt{x^2 - a^2}} - \int \frac{(x^2 - a^2)}{\sqrt{x^2 - a^2}} \cdot dx$$

$$I = x \sqrt{x^2 - a^2} - a^2 \underbrace{\int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx}_{\downarrow} - \int \sqrt{x^2 - a^2} \cdot dx$$

$$I = x \sqrt{x^2 - a^2} - a^2 \log |x + \sqrt{x^2 - a^2}| - I \quad \text{LHS } I = \text{Question}$$

$$\Rightarrow 2I = x \sqrt{x^2 - a^2} - a^2 \cdot \log |x + \sqrt{x^2 - a^2}|$$

$$\Rightarrow \boxed{I = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cdot \log |x + \sqrt{x^2 - a^2}| + C}$$

RMS

e.g. $I = \int \sqrt{5 - x^2} \cdot dx$ $\sqrt{a^2 - x^2} \rightarrow (a > x > -a)$

$$I = \int \sqrt{(5)^2 - x^2} \cdot dx \quad \quad \quad a = \sqrt{5} \quad \quad a^2 = 5$$

$$\boxed{\int \sqrt{a^2 - x^2} \cdot dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C}$$

$$I = \frac{x}{2} \sqrt{5 - x^2} + \frac{5}{2} \sin^{-1} \left(\frac{x}{\sqrt{5}} \right) + C \quad \checkmark$$

e.g. $I = \int \sqrt{3x^2 + 2x + 7} \cdot dx$

√Q (Lanzito)

Completing the Square method

$$I = \sqrt{3} \int \sqrt{x^2 + \frac{2x}{3} + \frac{7}{3}} \cdot dx$$

$$I = \sqrt{3} \int \sqrt{x^2 + \frac{2x}{3} + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2 + \frac{7}{3}} \cdot dx$$

$$I = \sqrt{3} \int \sqrt{\left(x + \frac{1}{3}\right)^2 + \frac{20}{9}} \cdot dx$$

~~I~~ $\underline{x + \frac{1}{3} = t} \Rightarrow \underline{dx = dt}$

$$I = \sqrt{3} \int \sqrt{t^2 + \left(\frac{2\sqrt{5}}{3}\right)^2} \cdot dt$$

$$I = \sqrt{3} \left\{ \frac{t}{2} \sqrt{t^2 + \frac{20}{9}} + \frac{10}{9} \log \left| t + \sqrt{t^2 + \frac{20}{9}} \right| + C \right.$$

$$\left. \int \sqrt{t^2 + a^2} \cdot dt = \frac{t}{2} \sqrt{t^2 + a^2} + \frac{a^2}{2} \log |t + \sqrt{t^2 + a^2}| + C \right\}$$

$$I = \sqrt{3} \left(\frac{x + \frac{1}{3}}{2} \cdot \sqrt{\left(x + \frac{1}{3}\right)^2 + \frac{20}{9}} + \frac{10}{9} \log \left| x + \frac{1}{3} + \sqrt{\left(x + \frac{1}{3}\right)^2 + \frac{20}{9}} \right| + C \right)$$

Exercise 7.7 $\int \sqrt{}$

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} 7$$

$$\begin{pmatrix} 3 \\ 9 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} 11$$

$$\int \sqrt{a^2 - x^2} \cdot dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

$$\int \sqrt{a^2 + x^2} \cdot dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log |x + \sqrt{a^2 + x^2}| + C$$

$$\int \sqrt{x^2 - a^2} \cdot dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + C$$

Q.2 $I = \int \sqrt{1 - 4x^2} \cdot dx = \int \sqrt{4(\frac{1}{4} - x^2)} \cdot dx$

$$I = 2 \int \sqrt{(\frac{1}{2})^2 - x^2} \cdot dx$$

$$a = \frac{1}{2}$$

$$a^2 = \frac{1}{4}$$

$$\int \sqrt{a^2 - x^2} \cdot dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

$$I = 2 \left(\frac{x}{2} \sqrt{\frac{1}{4} - x^2} + \frac{1}{4x^2} \sin^{-1} \left(\frac{x}{(\frac{1}{2})} \right) \right) + C$$

$$I = x \sqrt{\frac{1 - 4x^2}{4}} + \frac{1}{4} \sin^{-1}(2x) + C$$

$$I = \frac{x \sqrt{1 - 4x^2}}{2} + \frac{1}{4} \sin^{-1}(2x) + C$$

Q.3 $I = \int \sqrt{x^2 + 4x + 6} \cdot dx$

Completing Square method

$$I = \int \sqrt{x^2 + 4x + 2^2 - 2^2 + 6} \cdot dx$$

$$I = \int \sqrt{(x+2)^2 + 2} \cdot dx$$

$$I = \int \sqrt{\underbrace{(x+2)^2}_{(\bar{x})^2} + \underbrace{(\sqrt{2})^2}_{(\bar{a})^2}} \cdot dx$$

$$\int \sqrt{x^2 + a^2} \cdot dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$$

$$I = \frac{x+2}{2} \sqrt{(x+2)^2 + (\sqrt{2})^2} + \frac{(\sqrt{2})^2}{2} \log |(x+2) + \sqrt{(x+2)^2 + (\sqrt{2})^2}| + C$$

$$I = \frac{x+2}{2} \sqrt{x^2 + 4x + 6} + \log |(x+2) + \sqrt{x^2 + 4x + 6}| + C$$

Q.4 $I = \int \sqrt{x^2 + 4x + 1} \cdot dx$

Completing Square method

$$I = \int \sqrt{x^2 + 4x + 2^2 - 2^2 + 1} \cdot dx$$

$$I = \int \sqrt{(x+2)^2 - 3} \cdot dx$$

$$I = \int \frac{\sqrt{(x+2)^2 - (\sqrt{3})^2}}{x} \cdot dx$$

$$\boxed{\int \sqrt{x^2 - a^2} \cdot dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C}$$

$$I = \frac{x+2}{2} \sqrt{(x+2)^2 - (\sqrt{3})^2} - \frac{(\sqrt{3})^2}{2} \log \left| (x+2) + \sqrt{(x+2)^2 - (\sqrt{3})^2} \right|$$

$$\boxed{I = \frac{x+2}{2} \sqrt{x^2 + 4x + 1} - \frac{3}{2} \log \left| (x+2) + \sqrt{x^2 + 4x + 1} \right| + C}$$

$$\boxed{\text{Q.5}} \quad I = \int \sqrt{1 - 4x - x^2} \cdot dx$$

$$I = \int \sqrt{1 - (x^2 + 4x)} \cdot dx$$

Completing
Square method

$$I = \int \sqrt{1 - (x^2 + 4x + 2^2 - 2^2)} \cdot dx$$

$$I = \int \sqrt{1 - (x^2 + 4x + 2^2) + 4} \cdot dx$$

$$I = \int \sqrt{5 - (x+2)^2} \cdot dx$$

$$I = \int \sqrt{(\sqrt{5})^2 - (\underbrace{x+2}_x)^2} \cdot dx$$

$$\boxed{\int \sqrt{a^2 - x^2} \cdot dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C}$$

$$\boxed{I = \frac{x+2}{2} \sqrt{1 - 4x - x^2} + \frac{5}{2} \sin^{-1} \left(\frac{x+2}{\sqrt{5}} \right) + C}$$

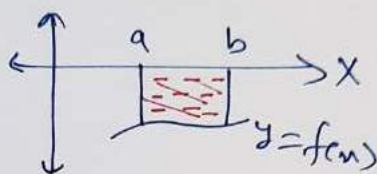
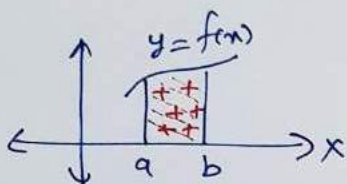
Definite Integral (निश्चित समाकलन)

$$\int_a^b f(x) \cdot dx$$

$x=a$ = lower limit of Definite Int.

$x=b$ = upper limit

Algebraic Area b/w Curve $y=f(x)$, $x=a$ line, $x=b$ line and x -axis.



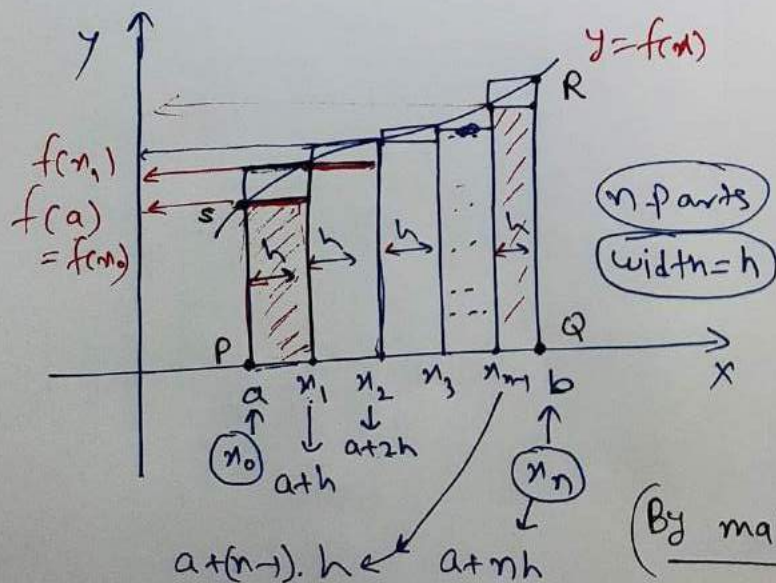
Definite Integral as the limit of a Sum

(योगफल की सीमा के रूप में निश्चित समाकलन)

$$\int_a^b f(x) \cdot dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h) \right]$$

(where $h = \frac{b-a}{n}$)

Let $f(x)$ is a continuous function in $[a, b]$, also $f(x) > 0$



$$\int_a^b f(x) \cdot dx = \text{Area (PQRSP)}$$

Now we will find this area with a different approach

(By making n rectangles)

upper rectangles
Areas to Sum

Exact Area of
curve

lower rectangles
Areas to Sum

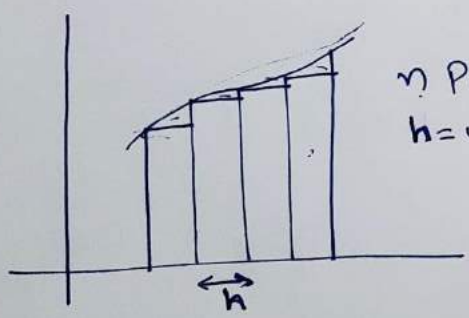
n less
h more

$n \rightarrow \infty$
 $h \rightarrow 0$

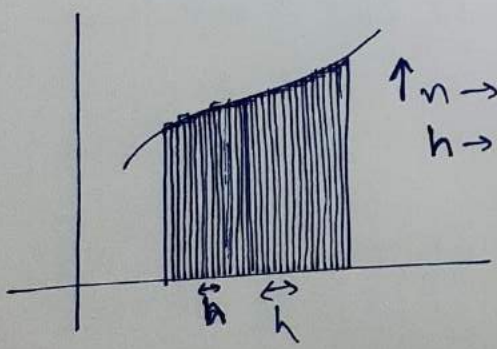
Indefinite Integral

$$\int f(x) \cdot dx = F(x) + c \rightarrow \text{!}$$

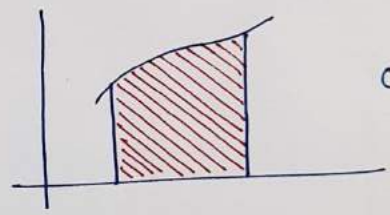
(Family of
curves
at a point)



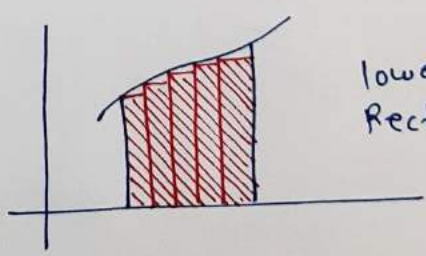
n Parts ↓
h = width



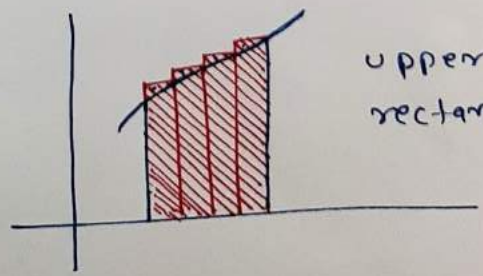
$\uparrow n \rightarrow \infty$
 $h \rightarrow 0$



curve



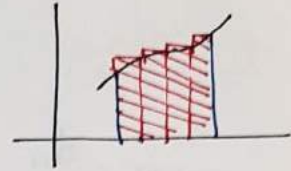
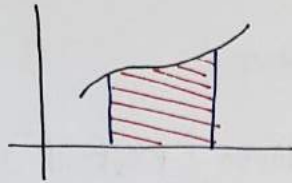
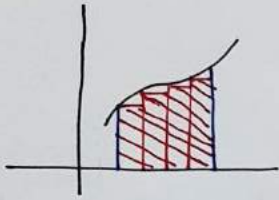
lower
Rectangles



upper
rectangles

$$\underbrace{S_n}_{\text{Areas of all lower rectangles}} < \text{Area(PQRSP)} < \underbrace{S_n}_{\text{Areas of all upper rectangles}}$$

$\int_a^b f(x) \cdot dx$



$$b = x_n = a + nh \Rightarrow b = a + n \cdot h$$

$$\Rightarrow b - a = n \cdot h \Rightarrow \boxed{h = \frac{b-a}{n}}$$

$S_n = \text{areas of all lower rectangles}$

$$= \underbrace{h \times f(x_0)} + \underbrace{h \times f(x_1)} + \underbrace{h \times f(x_2)} + \dots + h \times f(x_{n-1})$$

$$= h \cdot f(a) + h \times f(a+h) + h \cdot f(a+2h) + \dots + h \cdot f(a + (n-1)h)$$

$$= h \{ f(a) + f(a+h) + f(a+2h) + \dots + f(a + (n-1)h) \}$$

$S_n = \text{areas of all upper rectangles}$

$$= h \{ f(x_1) + f(x_2) + \dots + f(x_n) \}$$

$$= h \{ f(a+h) + f(a+2h) + \dots + f(a+nh) \}$$

$$\therefore \underline{S_n} < \text{Area(PQRSP)} < \overline{S_n}$$

If we increase 'n', then we can get both $\underline{S_n}$ & $\overline{S_n}$ close to Area(PQRSP).

If $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \underline{S_n} = \text{Area(PQRSP)} = \lim_{n \rightarrow \infty} \overline{S_n}$$

$$\int_a^b f(x) \cdot dx = \lim_{n \rightarrow \infty} (\underline{S_n})$$

$$\lim_{\substack{n \rightarrow \infty \\ h \rightarrow 0}}$$

$$= \lim_{\substack{n \rightarrow \infty \\ h \rightarrow 0}} (h) \{ f(a) + f(a+h) + \dots + f(a+(n-1) \cdot h) \}$$

$$\int_a^b f(x) \cdot dx = \lim_{n \rightarrow \infty} \left(\frac{b-a}{n} \right) \{ f(a) + f(a+h) + \dots + f(a+(n-1) \cdot h) \}$$

$$h = \frac{b-a}{n}$$

e.g. Find $\int_0^2 (x^2+1) \cdot dx$ as the limit of a sum.

$$\int_a^b f(x) \cdot dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ f(a) + f(a+h) + \dots + f(a+(n-1)h) \right\}$$

$$h = \frac{b-a}{n}$$

$a=0, b=2$, $h = \frac{2-0}{n} = \frac{2}{n}$, $f(x) = (x^2+1)$

$$\int_0^2 f(x) \cdot dx = (2-0) \cdot \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ f(0) + f(h) + f(2h) + \dots + f((n-1)h) \right\}$$

$$\int_0^2 (x^2+1) \cdot dx = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ f(0) + f\left(\frac{2}{n}\right) + f\left(\frac{4}{n}\right) + \dots + f\left((n-1) \cdot \frac{2}{n}\right) \right\}$$

$$= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \left[\frac{0^2+1}{n} \right] + \left[\frac{\left(\frac{2}{n}\right)^2+1}{n} \right] + \left[\frac{\left(\frac{4}{n}\right)^2+1}{n} \right] + \dots + \left[\frac{\left(\frac{2(n-1)}{n}\right)^2+1}{n} \right] \right\}$$

$$= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \left(\underbrace{1+1+1+\dots+1}_{n \text{ times}} \right) + \left(\frac{0^2}{n} + \frac{\left(\frac{2}{n}\right)^2}{n} + \frac{\left(\frac{4}{n}\right)^2}{n} + \dots + \frac{\left(\frac{2(n-1)}{n}\right)^2}{n} \right) \right\}$$

$$= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ n + \left(\frac{2}{n}\right)^2 \cdot \left[(1)^2 + (2)^2 + (3)^2 + \dots + (n-1)^2 \right] \right\}$$

$$= 2 \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) \left\{ \underbrace{n}_{\frac{1}{n}} + \underbrace{\left(\frac{2}{n}\right)^2 \cdot \left[\frac{(n-1) \cdot n \cdot (2n-1)}{6} \right]}_{\frac{2}{n^3} \cdot (3)} \right\}$$

$$= 2 \lim_{n \rightarrow \infty} \left\{ \frac{n}{n} + \frac{2 \cdot (n-1) \cdot n \cdot (2n-1)}{(n^3) \cdot (3)} \right\}$$

$$\int_0^2 f(x) \cdot dx = 2 \lim_{n \rightarrow \infty} \left\{ \frac{n}{n} + \frac{2(n-1)(n)(2n-1)}{3 \cdot n^3} \right\}$$

$$= 2 \lim_{n \rightarrow \infty} \left\{ 1 + \frac{2}{3} \cdot \frac{(n-1)}{(n)} \cdot \frac{n}{n} \cdot \frac{(2n-1)}{(n)} \right\}$$

$$n^3 = n \cdot n \cdot n$$

$$= 2 \lim_{n \rightarrow \infty} \left\{ 1 + \frac{2}{3} \left(1 - \frac{1}{n}\right) \cdot \left(2 - \frac{1}{n}\right) \right\}$$

$$= 2 \left\{ 1 + \frac{2}{3} (1-0)(2-0) \right\} \quad \left(\text{when } n \rightarrow \infty, \frac{1}{n} \rightarrow 0 \right)$$

$$= 2 \left(1 + \frac{4}{3} \right) = 2 \times \frac{7}{3} = \frac{14}{3}$$

e.g. Evaluate $\int_0^2 e^x \cdot dx$ as the limit of a sum.

$$\int_a^b f(x) \cdot dx = (b-a) \cdot \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ f(a) + f(a+h) + \dots + f(a+(n-1)h) \right\}$$

$$\underline{a=0}, \underline{b=2}, \underline{f(x)=e^x}, \quad h = \frac{b-a}{n} = \frac{2}{n} \quad (\text{cancel } h)$$

$$\int_0^2 e^x \cdot dx = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ f(0) + f\left(\frac{2}{n}\right) + f\left(\frac{4}{n}\right) + \dots + f\left(\frac{2(n-1)}{n}\right) \right\}$$

$$= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \underline{e^0} + \underline{e^{\frac{2}{n}}} + \underline{e^{\frac{4}{n}}} + \dots + \underline{e^{\frac{2(n-1)}{n}}} \right\}$$

Sum of G.P. Geometric Progression

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \left\{ \underline{a=e^0=1}, \underline{r=e^{\frac{2}{n}}} \right\}$$

$$\int_0^2 e^n \cdot dn = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ 1 \cdot \frac{(e^{\frac{2}{n}})^n - 1}{(e^{2/n}) - 1} \right\}$$

$$\frac{2}{\frac{2}{n}} = n$$

$$= \cancel{2} (e^2 - 1) \cdot \lim_{n \rightarrow \infty} \left(\frac{2}{n} \right) \cdot \frac{1}{e^{2/n} - 1}$$

$$= (e^2 - 1) \cdot \left(\lim_{h \rightarrow 0} \frac{h}{e^h - 1} \right)$$

$$= (e^2 - 1) \cdot 1$$

$$\int_0^2 e^n \cdot dn = (e^2 - 1)$$

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

Standard form

Exercise 7.8

Definite Integral as the limit of a Sum

Exercise 7.8 start करने से पहले इन Formulas को अच्छे से याद कर लें

$$\int_a^b f(x) \cdot dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)]$$

where $h = \frac{b-a}{n}$

$$\int_a^b f(x) \cdot dx = \lim_{h \rightarrow 0} h [f(a) + f(a+h) + \dots + f(a+(n-1)h)]$$

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

$$1+2+3+\dots+(n-1) = \frac{(n-1)n}{2}$$

$$1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^2+2^2+\dots+(n-1)^2 = \frac{(n-1)n(2n-1)}{6}$$

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1 \left(\frac{0}{0} \text{ Form} \right)$$

Exercise 7.8 / class 12

Q.2 $\int_0^5 (x+1) \cdot dx$

$$h = \frac{b-a}{n}$$

$a=0$
 $b=5$
 $n = \frac{5}{h}$
 $h = \frac{5}{n}$

$$\int_a^b f(x) \cdot dx = \lim_{h \rightarrow 0} h [f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)]$$

$a+h=h$

$$\int_0^5 (x+1) \cdot dx = \lim_{h \rightarrow 0} h [f(0) + f(h) + f(2h) + \dots + f((n-1)h)]$$

$$= \lim_{h \rightarrow 0} h \left[\underbrace{(0+1)}_{\text{1st term}} + \underbrace{(h+1)}_{\text{2nd term}} + \underbrace{(2h+1)}_{\text{3rd term}} + \dots + \underbrace{((n-1)h+1)}_{\text{n-th term}} \right]$$

$n\text{-terms}$

$$\begin{aligned}
 \int_0^5 (x+1) \cdot dx &= \lim_{h \rightarrow 0} h \left[\underbrace{(1+1+1+\dots+1)}_{n\text{-times}} + \underbrace{(0+h+2h+\dots+(n-1)h)}_{(n-1)\text{ terms}} \right] \\
 &= \lim_{h \rightarrow 0} h \left[(n) + h (1+2+3+\dots+(n-1)) \right] \\
 &= \lim_{h \rightarrow 0} h \left[\underline{(n)} + h \left\{ \frac{(n-1) \cdot n}{2} \right\} \right] \quad \text{Formula 9} \\
 &= \lim_{h \rightarrow 0} h \left[\frac{5}{h} + h \left\{ \frac{(\frac{5}{h}-1) \cdot \frac{5}{h}}{2} \right\} \right] \\
 &= \lim_{h \rightarrow 0} \left[5 + h \cdot \frac{(\frac{5-h}{h}) \cdot 5}{2} \right] \\
 &= \lim_{h \rightarrow 0} \left[5 + \frac{(5-h) \cdot 5}{2} \right] = 5 + \frac{(5-0) \cdot 5}{2} = 5 + \frac{25}{2}
 \end{aligned}$$

$$\begin{aligned}
 h &= \frac{5}{n} \\
 nh &= 5 \\
 n &= \frac{5}{h}
 \end{aligned}$$

$$\boxed{\int_0^5 (x+1) \cdot dx = \frac{35}{2}} \quad \checkmark$$

[Q.4] $\int_1^4 (x^2 - x) \cdot dx$

$$\begin{aligned}
 a &= 1 \\
 b &= 4
 \end{aligned}$$

$$\begin{aligned}
 h &= \frac{b-a}{n} \\
 nh &= b-a \quad \rightarrow \boxed{nh=3}
 \end{aligned}$$

$$\begin{aligned}
 \int_a^b f(x) \cdot dx &= \lim_{h \rightarrow 0} h \cdot [f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)] \\
 \int_1^4 (x^2 - x) \cdot dx &= \lim_{h \rightarrow 0} h \cdot [f(1) + f(1+h) + f(1+2h) + \dots + f(1+(n-1)h)] \\
 &= \lim_{h \rightarrow 0} h \left[\underline{(1^2 - 1)} + \underline{((1+h)^2 - (1+h))} + \underline{((1+2h)^2 - (1+2h))} \right. \\
 &\quad \left. \dots + \underline{((1+(n-1)h)^2 - (1+(n-1)h))} \right]
 \end{aligned}$$

$$\int_1^4 (x^2 - x) \cdot dx = \lim_{h \rightarrow 0} h \left[\frac{(0)}{1} + \frac{h^2 + h}{2} + \frac{(2h)^2 + 2h}{3} + \dots + \frac{((n-1)h)^2 + (n-1)h}{n} \right]$$

$$= \lim_{h \rightarrow 0} h \left[\left\{ \frac{h^2}{1} + \frac{(2h)^2}{2} + \frac{(3h)^2}{3} + \dots + \frac{((n-1)h)^2}{n} \right\} + \left\{ h + 2h + \dots + (n-1)h \right\} \right]$$

$$= \lim_{h \rightarrow 0} h \left[h^2 \left(\frac{1^2 + 2^2 + 3^2 + \dots + (n-1)^2}{1} \right) + h \left(1 + 2 + 3 + \dots + (n-1) \right) \right]$$

$$= \lim_{h \rightarrow 0} h \left[h^2 \left(\frac{(n-1) \cdot n \cdot (2n-1)}{6} \right) + h \left(\frac{(n-1) \cdot n}{2} \right) \right]$$

$$\therefore nh = 3 \Rightarrow n = \frac{3}{h}$$

$$= \lim_{h \rightarrow 0} h \left[h^3 \frac{\left(\frac{3}{h} - 1 \right) \cdot \frac{3}{h} \cdot \left(2 \times \frac{3}{h} - 1 \right)}{6} + h^2 \frac{\left(\frac{3}{h} - 1 \right) \cdot \frac{3}{h}}{2} \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{\cancel{h^3} \cdot \left(\frac{3-h}{h} \right) \cdot \frac{3}{h} \cdot \frac{(6-h)}{h}}{6} + \frac{\cancel{h^2} \cdot \left(\frac{3-h}{h} \right) \cdot \frac{3}{h}}{2} \right]$$

$(h \rightarrow 0)$ put

$$= \frac{(3-0) \cdot (3) \cdot (6-0)}{6} + \frac{(3-0) \cdot 3}{2}$$

$$= 9 + \frac{9}{2} = \frac{27}{2}$$

Q.6

$$\int_0^4 (x + e^{2x}) \cdot dx$$

$$\begin{matrix} a=0 \\ b=4 \end{matrix}$$

$$h = \frac{b-a}{n} \quad nh = b-a \quad \boxed{nh=4}$$

$$\int_a^b f(x) \cdot dx = \lim_{h \rightarrow 0} h \left[f(a) + f(a+h) + \dots + f(a+(n-1)h) \right]$$

$$\int_0^4 (x + e^{2x}) \cdot dx = \lim_{h \rightarrow 0} h \left[f(0) + f(h) + f(2h) + \dots + f((n-1)h) \right]$$

$$= \lim_{h \rightarrow 0} h \cdot \left[(1) + (h + e^{2h}) + (2h + e^{4h}) + \dots + ((n-1)h + e^{2(n-1) \cdot h}) \right]$$

$$= \lim_{h \rightarrow 0} h \left[\overbrace{h + 2h + \dots + (n-1)h}^{(n-1) \text{ terms.}} + \underbrace{\left\{ 1 + e^{2h} + e^{4h} + \dots + e^{2(n-1) \cdot h} \right\}}_{n\text{-terms}} \right]$$

$$\text{Sum of G.P.} = \frac{a(r^n - 1)}{(r - 1)}$$

Here $a=1$

$$\text{C.R.} = r = \frac{e^{2h}}{1} = e^{2h}$$

$$\boxed{nh=4} \text{ Starting.}$$

$$h \{ 1 + 2 + \dots + (n-1) \}$$
$$= h \cdot \left\{ \frac{(n-1) \cdot n}{2} \right\}$$

$$= \frac{h(n-1) \cdot n}{2}$$

$$= \frac{2(4) \cdot (n-1)}{2}$$

$$= 4 \cdot 2 \left(\frac{4}{n} - 1 \right) = 2 \left(\frac{4 - h}{h} \right)$$

$$\int_0^4 (x + e^{2x}) \cdot dx = \lim_{h \rightarrow 0} h \left[\underbrace{\frac{2(4-h)}{h}} + \frac{e^8 - 1}{e^{2h} - 1} \right]$$

$$= \lim_{h \rightarrow 0} \left[2(4-h) + \frac{(e^8 - 1) \cdot h \times 2}{2x(e^{2h} - 1)} \right] \rightarrow \frac{0}{e^0 - 1}$$

Standard form

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

$$\lim_{h \rightarrow 0} \frac{e^{2h} - 1}{2h} = 1$$

$$\lim_{h \rightarrow 0} \left(\frac{2h}{e^{2h} - 1} \right) = 1$$

$$= \frac{0}{1-1} = \frac{0}{0} \text{ Form}$$

$$\int_0^4 (x + e^{2x}) \cdot dx = 2(4-0) + \frac{e^8 - 1}{2} \times (1)$$

$$= 8 + \frac{e^8 - 1}{2}$$

$$= \frac{16 + e^8 - 1}{2} = \frac{15 + e^8}{2}$$

Definite Integral

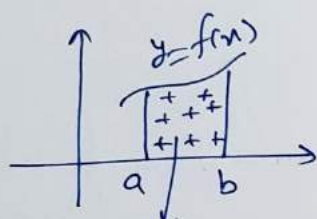
निश्चित समाकलन

$$\int_a^b f(x) \cdot dx$$

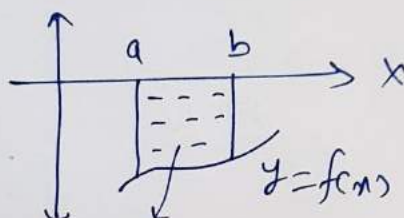
$x=b$ = upper limit of Definite Integral.

$x=a$ = lower limit of Definite integral

Algebraic Area between Curve $y=f(x)$,
line $x=a$, line $x=b$ and y-axis.



$$\int_a^b f(x) \cdot dx \oplus$$



$$\int_a^b f(x) \cdot dx \ominus$$

How to evaluate $\int_a^b f(x) \cdot dx$?

$$\int f(x) \cdot dx = F(x) + c$$

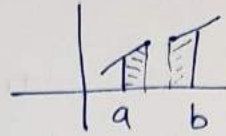
$$\int_a^b f(x) \cdot dx = [F(x)]_a^b = F(b) - F(a)$$

$$[F(x) + c]_a^b = (F(b) + \cancel{c}) - (F(a) + \cancel{c})$$

Note: ① $\int_a^b f(x) \cdot dx = \int_a^b f(t) \cdot dt = \int_a^b f(y) \cdot dy$
 $= F(b) - F(a)$

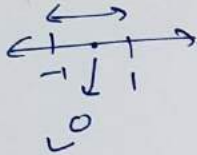
② In $\int_a^b f(x) \cdot dx$, f should be continuous in $[a, b]$.

$\int_{-1}^1 \left(\frac{1}{x} \right) \cdot dx$ → wrong & Not defined



$\frac{1}{x} \neq 0$

$\frac{1}{0} = \infty$



e.g. Evaluate $\int_2^5 \underline{x^2 \cdot dx}$

$= \left[\frac{x^3}{3} \right]_2^5$

$$\boxed{\int_a^b f(x) \cdot dx = [F(x)]_a^b = F(b) - F(a)}$$

$= \left(\frac{5^3}{3} \right) - \left(\frac{2^3}{3} \right) = \frac{125}{3} - \frac{8}{3} = \frac{117}{3} = \underline{\underline{\text{Constant}}}$

e.g. $I = \int_0^{\frac{\pi}{4}} \underline{\sin^3 2t \cdot \cos 2t \cdot dt}$ $= \left[\frac{\sin^4 2t}{8} \right]_0^{\frac{\pi}{4}}$

$I = \int \underline{\sin^3 2t \cdot \cos 2t \cdot dt}$

Substitution

$\sin 2t = x$ ✓

$\Rightarrow \cos 2t \cdot 2 dt = dx$

$\Rightarrow \cos 2t dt = \frac{dx}{2}$ ✓

$I = \int x^3 \cdot \frac{dx}{2} = \frac{1}{2} \left(\frac{x^4}{4} \right) = \frac{x^4}{8} + C = \frac{\sin^4 2t}{8} + C$

$= \left(\frac{\sin^4 \left(2 \times \frac{\pi}{4} \right)}{8} \right) - \left(\frac{\sin^4 0}{8} \right)$

$= \frac{1^4}{8} = \frac{1}{8}$ ✓

Exercise 7.9 → Definite Integration

Indefinite

$$\int f(x) \cdot dx = F(x) + C$$

$$\int_a^b f(x) \cdot dx = [F(x)]_a^b$$

$$= F(b) - F(a)$$

Q.1 $I = \int_{-1}^1 (x^2 + 1) \cdot dx = \left[\frac{x^2}{2} + x \right]_{-1}^1 = \left[\frac{1^2}{2} + 1 \right] - \left[\frac{(-1)^2}{2} - 1 \right]$

$$= \frac{1}{2} + 1 - \frac{1}{2} + 1 = 2$$

Q.2 $I = \int_2^3 \left(\frac{1}{x} \right) \cdot dx = [\log |x|]_2^3 = \log 3 - \log 2$

$$= \log \frac{3}{2}$$

Q.3 $I = \int_1^2 (4x^3 - 5x^2 + 6x + 9) \cdot dx = \left[\frac{4x^4}{4} - 5 \cdot \frac{x^3}{3} + \frac{6x^2}{2} + 9x \right]_1^2$

$$= \left[2^4 - \frac{5}{3} \cdot 2^3 + 3(2^2) + 9 \times 2 \right] - \left[1 - \frac{5}{3} + 3 + 9 \right]$$

$$= \frac{64}{3}$$

Q.4 $I = \int_0^{\frac{\pi}{4}} \sin 2x \cdot dx$

$\int \sin(ax+b) \cdot dx = -\frac{\cos(ax+b)}{a} + C$

$$I = \left[-\frac{\cos 2x}{2} \right]_0^{\frac{\pi}{4}} = \left[-\frac{\cos 2 \times \frac{\pi}{4}}{2} \right] - \left[-\frac{\cos 2(0)}{2} \right]$$

$$= 0 + \frac{1}{2} = \frac{1}{2}$$

$\cos 0 = 1$
 $\cos \frac{\pi}{2} = 0$

Q.5
$$I = \int_0^{\frac{\pi}{2}} \cos 2x \cdot dx = \left[\frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}} = \left[\frac{\sin \pi}{2} \right] - \left[\frac{\sin 0}{2} \right]$$

$$= 0 - 0 = 0$$

$\sin \pi = 0$
 $\sin 0 = 0$

Q.6
$$I = \int_4^5 e^x \cdot dx = (e^x)_4^5 = e^5 - e^4 = \underline{\underline{e^4(e-1)}}$$

Q.7
$$I = \int_0^{\frac{\pi}{4}} \tan x \cdot dx = \left[\log |\sec x| \right]_0^{\frac{\pi}{4}}$$

$$= \log \left| \sec \frac{\pi}{4} \right| - \log |\sec 0|$$

$$= \log |\sqrt{2}| - \underbrace{\log(1)}_0 = \log(2^{\frac{1}{2}}) - 0$$

$$= \frac{1}{2} \log 2$$

$\cos 0 = 1$
 $\log m^n = n \cdot \log m$

Q.8
$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \cdot dx$$

$\frac{\pi}{6} = 30^\circ, \quad \frac{\pi}{4} = 45^\circ$
 $\tan 30^\circ = \frac{1}{\sqrt{3}}$
 $\sin \frac{\pi}{6} = \frac{1}{2}$

$$I = \left[\log |\operatorname{cosec} x - \cot x| \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$I = \log \left| \operatorname{cosec} \frac{\pi}{4} - \cot \frac{\pi}{4} \right| - \log \left| \operatorname{cosec} \frac{\pi}{6} - \cot \frac{\pi}{6} \right|$$

$$I = \log \left| \frac{\sqrt{2} - 1}{2 - \sqrt{3}} \right|$$

$$\boxed{\text{Q.9}} \quad I = \int_0^1 \left(\frac{dx}{\sqrt{1-x^2}} \right) = (\sin^{-1}x)_0^1 = \sin^{-1}(1) - \sin^{-1}(0) \\ = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\boxed{\text{Q.10}} \quad I = \int_0^1 \left(\frac{dx}{1+x^2} \right) = (\tan^{-1}x)_0^1 = \tan^{-1}(1) - \tan^{-1}(0) \\ = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\boxed{\text{Q.11}} \quad I = \int_2^3 \left(\frac{dx}{x^2-1^2} \right) = \left[\frac{1}{2(1)} \cdot \log \left| \frac{x-1}{x+1} \right| \right]_2^3 \\ = \left(\frac{1}{2} \log \left| \frac{2}{3} \right| \right) - \left(\frac{1}{2} \log \left| \frac{1}{3} \right| \right) \quad \log m - \log n = \log \frac{m}{n} \\ = \frac{1}{2} \left(\log \left| \frac{2}{3} \right| - \log \left| \frac{1}{3} \right| \right) = \frac{1}{2} \log \left| \frac{\frac{2}{3}}{\frac{1}{3}} \right| \\ = \frac{1}{2} \log \left| \frac{2}{1} \right| = \frac{1}{2} \log 2$$

$$\boxed{\text{Q.12}} \quad I = \int_0^{\frac{\pi}{2}} \cos^2 x \cdot dx \\ I = \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2} \right) \cdot dx = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{\cos 2x}{2} \right) \cdot dx \\ I = \left(\frac{1}{2}x + \frac{\sin 2x}{4} \right)_0^{\frac{\pi}{2}} = \left(\frac{1}{2} \cdot \frac{\pi}{2} + \frac{\sin \pi}{4} \right) - (0+0) \\ = \frac{\pi}{4} + 0 = \frac{\pi}{4}$$

Exercise 7.9 → Definite integration

$$\int_a^b f(x) \cdot dx = \left[F(x) \right]_a^b = \underline{F(b) - F(a)}$$

(Formula, by Subst., Partial Fraction, by Parts)

Q.13

$$I = \int_2^3 \frac{x}{x^2+1} \cdot dx$$

u.v

$$I = \left[\frac{1}{2} \log |x^2+1| \right]_2^3$$

$$\int \frac{x}{x^2+1} \cdot dx \rightarrow t$$

$$\text{let } x^2+1 = t$$

$$\Rightarrow 2x \cdot dx = dt$$

$$\Rightarrow x \cdot dx = \frac{dt}{2}$$

$$\frac{1}{2} \int \frac{dt}{t}$$

$$= \frac{1}{2} \log |t| + c$$

$$= \frac{1}{2} \log |x^2+1| + c$$

$$I = \frac{1}{2} \log(10) - \frac{1}{2} \log(5)$$

$$I = \frac{1}{2} \{ \log 10 - \log 5 \} = \frac{1}{2} \log \left(\frac{10}{5} \right) = \frac{1}{2} \log(2)$$

$$\log m - \log n = \log \frac{m}{n}$$

Q.14 $I = \int_0^1 \frac{2x+3}{5x^2+1} \cdot dx$

$$\int \frac{2x+3}{5x^2+1} \cdot dx = \int \frac{2x}{5x^2+1} \cdot dx \rightarrow I_1$$

$$I_1 = \int \frac{2x}{5x^2+1} \cdot dx$$

$$+ \int \frac{3}{5x^2+1} \cdot dx \rightarrow I_2$$

$$\text{let } 5x^2+1 = t \rightarrow I_1 = \frac{1}{5} \int \frac{dt}{t}$$

$$\Rightarrow 10x \cdot dx = dt$$

$$\Rightarrow \underline{2x \cdot dx} = \left(\frac{dt}{5} \right)$$

$$\underline{I_1 = \frac{1}{5} \log |t| = \frac{1}{5} \log |5x^2+1|}$$

$$I_2 = \int \frac{3}{5x^2+1} \cdot dx = \frac{3}{5} \int \frac{1}{x^2 + \frac{1}{5}} \cdot dx$$

$$a^2 = \frac{1}{5}$$

$$a = \frac{1}{\sqrt{5}}$$

$$\left(\int \frac{1}{x^2 + a^2} \cdot dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C \right)$$

$$I_2 = \frac{3}{5} \cdot \left(\frac{1}{\sqrt{5}} \right) \cdot \tan^{-1} \left(\frac{x}{\frac{1}{\sqrt{5}}} \right)$$

$$I_2 = \frac{3\sqrt{5}}{5} \tan^{-1}(\sqrt{5}x) \checkmark$$

$$\therefore \int \frac{(2x+3)}{5x^2+1} \cdot dx = \frac{I_1}{1} + \frac{I_2}{a} = \frac{1}{5} \log(x^2+1) + \frac{3\sqrt{5}}{5} \tan^{-1}(\sqrt{5}x)$$

$$\begin{aligned} \int_0^1 \left(\frac{2x+3}{5x^2+1} \right) \cdot dx &= \left[\frac{1}{5} \log(x^2+1) + \left(\frac{3\sqrt{5}}{5} \right) \tan^{-1}(\sqrt{5}x) \right]_0^1 \\ &= \left[\frac{1}{5} \log(1+1) + \frac{3}{\sqrt{5}} \tan^{-1}(\sqrt{5}) \right] - \left[\frac{1}{5} \log(1) + \frac{3}{\sqrt{5}} \tan^{-1}(0) \right] \\ &= \frac{1}{5} \log 2 + \frac{3}{\sqrt{5}} \tan^{-1}(\sqrt{5}) \checkmark \end{aligned}$$

[Q.15]

$$I = \int_0^1 \frac{x \cdot e^{x^2}}{1} \cdot dx$$

$$\int x \cdot e^{x^2} \cdot dx$$

$$x^2 = t$$

$$2x \cdot dx = dt$$

$$x \cdot dx = \frac{dt}{2} \checkmark$$

$$I = \left[\frac{1}{2} e^{x^2} \right]_0^1$$

$$I = \left[\frac{1}{2} e^1 - \frac{1}{2} e^0 \right]$$

$$I = \frac{1}{2} e - \frac{1}{2} = \frac{1}{2} (e - 1) \checkmark$$

$$\begin{aligned} &\frac{1}{2} \int e^t dt \\ &= \frac{1}{2} e^t \\ &= \frac{1}{2} e^{x^2} \end{aligned}$$

[Q.16] $I = \int_1^2 \frac{5x^2}{x^2+4x+3} \cdot dx$

(Degree of Nr = Degree of Dr.) $\rightarrow$ Long Division
 (2) (2)

$$\frac{5x^2}{x^2+4x+3} = 5 - \frac{20x+15}{x^2+4x+3}$$

Partial Fraction,

$$\begin{array}{r} x^2+4x+3 \overline{) 5x^2} \quad (5 \\ \underline{5x^2+20x+15} \\ -20x-15 \\ \underline{-(20x+15)} \\ 0 \end{array}$$

$$\frac{20x+15}{x^2+4x+3} = \frac{20x+15}{(x+1)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+3)}$$

Proper method

$$20x+15 = A(x+3) + B(x+1)$$

Comparison, $(x) \rightarrow 20 = A + B$

Constant $\rightarrow 15 = 3A + B$

$$5 = -2A$$

$$A = -\frac{5}{2}$$

$$B = \frac{45}{2}$$

$$\frac{5x^2}{x^2+4x+3} = 5 - \left[\frac{-5}{2(x+1)} + \frac{45}{2(x+3)} \right]$$

I

$$I = \int_1^2 \frac{5x^2}{x^2+4x+3} \cdot dx = \int_1^2 \left[5 + \frac{5}{2(x+1)} - \frac{45}{2(x+3)} \right] \cdot dx$$

$$I = \left[5x + \frac{5}{2} \log|x+1| - \frac{45}{2} \log|x+3| \right]_1^2$$

$$I = \left(\underline{10} + \frac{5}{2} \log(3) - \frac{45}{2} \log(5) \right) - \left(\underline{5} + \frac{5}{2} \log(2) - \frac{45}{2} \log(4) \right)$$

$$I = 5 + \frac{5}{2} \left\{ \log 3 - \log 2 \right\} + \frac{45}{2} \left\{ -\log 5 + \log 4 \right\}$$

$$I = 5 - \frac{5}{2} \left\{ \log 2 - \log 3 + 9(\log 5 - \log 4) \right\}$$

$$I = 5 - \frac{5}{2} \left\{ \log \frac{2}{3} + 9 \cdot \log \frac{5}{4} \right\}$$

$$I = 5 - \frac{5}{2} \left\{ -\log \frac{3}{2} + 9 \cdot \log \frac{5}{4} \right\}$$

Q.17 $I = \int_0^{\frac{\pi}{4}} (2 \sec^2 x + x^3 + 2) \cdot dx$

$$I = \left(2 \tan x + \frac{x^4}{4} + 2x \right) \Big|_0^{\frac{\pi}{4}}$$

$$I = \left(2 \tan \frac{\pi}{4} + \frac{\left(\frac{\pi}{4}\right)^4}{4} + 2 \times \frac{\pi}{4} \right) - 0$$

$$I = 2 + \frac{\pi^4}{1024} + \frac{\pi}{2}$$

$$\boxed{\text{Q.18}} \quad I = \int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) \cdot dx$$

$$I = - \int_0^{\pi} \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) \cdot dx$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\frac{x}{2} \times 2 = x$$

$$I = - \int_0^{\pi} (\cos x) \cdot dx$$

$$I = - (\sin x)_0^{\pi} = - (\sin \pi - \sin 0) = 0$$

$$\boxed{\text{Q.19}} \quad I = \int_0^2 \frac{6x+3}{x^2+4} \cdot dx$$

$$4 = 2^2$$

$$\int \frac{3}{x^2+4} \cdot dx$$

$$= 3 \left(\frac{1}{2} \tan^{-1} \frac{x}{2} \right)$$

$$= \frac{3}{2} \tan^{-1} \frac{x}{2}$$

$$\int \frac{6x}{x^2+4} \cdot dx$$

$$x^2+4 = t$$

$$2x \cdot dx = dt$$

$$3 \int \frac{dt}{t} = 3 \log |t|$$

$$= 3 \log |x^2+4|$$

$$I = \int_0^2 \left(\frac{6x}{x^2+4} + \frac{3}{x^2+4} \right) \cdot dx$$

$$I = \left[3 \log |x^2+4| + \frac{3}{2} \tan^{-1} \frac{x}{2} \right]_0^2$$

$$I = \left[3 \log(8) + \frac{3}{2} \tan^{-1}(1) \right] - \left(3 \log(4) + \frac{3}{2} \tan^{-1}(0) \right)$$

$$I = 3 \log \left(\frac{8}{4} \right) + \frac{3}{2} \times \frac{\pi}{4} = 3 \log(2) + \frac{3\pi}{8}$$

Q20 $I = \int_0^1 \left(\underbrace{x \cdot e^x}_{\text{By Parts}} + \underbrace{\sin \frac{\pi}{4} x}_{\text{Formula}} \right) \cdot dx$

$$I = \int_0^1 x \cdot e^x \cdot dx + \int_0^1 \sin \frac{\pi}{4} x \cdot dx$$

$\int x \cdot e^x \cdot dx$ **ILATE** = $\int \underbrace{x}_I \cdot \underbrace{e^x}_{II} \cdot dx$
(Int. by Parts)

$$\boxed{\int I \cdot II = I \int II - \int (I' \cdot \int II)}$$

$$= x \cdot e^x - \int e^x \cdot dx$$

$$= x \cdot e^x - e^x$$

$$\int \sin \frac{\pi}{4} x \cdot dx = \frac{-\cos\left(\frac{\pi}{4} x\right)}{\frac{\pi}{4}} = \frac{-4 \cos\left(\frac{\pi}{4} x\right)}{\pi}$$

$$I = \int_0^1 x e^x \cdot dx + \int_0^1 \sin \frac{\pi}{4} x \cdot dx$$

$$I = \left(x e^x - e^x \right)_0^1 + \left(\frac{-4 \cos\left(\frac{\pi}{4} x\right)}{\pi} \right)_0^1$$

$$I = \left(\cancel{1} - \cancel{e} \right) - (0 - 1) + \left(\frac{-4}{\pi} \cos\left(\frac{\pi}{4}\right) \right) - \left(\frac{-4}{\pi} \cos 0 \right)$$

$$I = 1 - \frac{4}{\pi} \cdot \frac{1}{\sqrt{2}} + \frac{4}{\pi}$$

$$\boxed{I = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}}$$

$$\boxed{\text{Q.21}} \quad I = \int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \left(\tan^{-1} x \right)_1^{\sqrt{3}}$$

$$I = \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$$

$$I = \frac{\pi}{3} - \cancel{\frac{\pi}{4}} = \frac{\pi}{12} \quad \checkmark$$

option 2

$\boxed{\text{Q.22}}$

$$I = \int_0^{2/3} \frac{dx}{4+9x^2} = \frac{1}{9} \int_0^{2/3} \frac{dx}{\frac{4}{9} + x^2}$$

$$I = \frac{1}{9} \left(\frac{1}{a} \tan^{-1} \frac{x}{a} \right)_0^{2/3}$$

$$a = \frac{2}{3}$$

$$I = \frac{1}{9} \left(\frac{3}{2} \tan^{-1} \left(\frac{3x}{2} \right) \right)_0^{2/3}$$

$$\tan^{-1}(1) = \frac{\pi}{4}$$

$$= \frac{1}{9} \left(\frac{3}{2} \tan^{-1} \left(\frac{3}{2} \times \frac{2}{3} \right) - \left(\frac{3}{2} \tan^{-1}(0) \right) \right)$$

$$I = \frac{1}{9} \left(\frac{3}{2} \times \frac{\pi}{4} \right) = \left(\frac{3\pi}{8} \right) \times \frac{1}{9} = \frac{\pi}{24} \quad \checkmark$$

(C)

Definite Integration by Substitution (प्रतिस्थापन द्वारा निश्चित समाकलन)

Old method:

$$\int_a^b f(x) \cdot dx \quad \xrightarrow{x \rightarrow t} \quad \int g(t) \cdot dt$$

$$\downarrow$$

$$F(x) + C \quad \leftarrow \quad G(t) + C$$

$$\swarrow \quad \searrow$$

$$\boxed{F(b) - F(a)} \quad \leftarrow \quad \boxed{G(t) + C} \xrightarrow{t \rightarrow x}$$

New method:

$$\int_a^b f(x) \cdot dx \quad \xrightarrow{x \rightarrow t} \quad \int_c^d g(t) \cdot dt$$

$b \rightarrow \text{old upper limit}$
 $a \rightarrow \text{old lower limit}$
 $\text{New upper limit} = d$
 $\text{New lower limit} = c$

$$\downarrow$$

$$\boxed{G(d) - G(c)} \quad \leftarrow \quad [G(t)]_c^d$$

e.g. $I = \int_2^3 \frac{x}{x^2+1} \cdot dx$

old method

$$\int \frac{x}{x^2+1} \cdot dx \quad \rightarrow \quad \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log(t)$$

$$= \frac{1}{2} \log(x^2+1)$$

$$I = \int_2^3 \frac{x}{x^2+1} \cdot dx$$

$$I = \left[\frac{1}{2} \log(x^2+1) \right]_2^3$$

$$= \frac{1}{2} \{ \log 10 - \log 5 \} = \frac{1}{2} \log \frac{10}{5} = \frac{1}{2} \log 2$$

By New method

$$x^2+1=t$$

$$I = \int_2^3 \frac{x}{x^2+1} \cdot dx$$

$$\text{let } x^2+1=t$$

$$\Rightarrow 2x \cdot dx = dt$$

$$\Rightarrow x dx = \frac{dt}{2}$$

$$I = \frac{1}{2} \int_5^{10} \frac{dt}{t} = \frac{1}{2} \left(\log|t| \right)_5^{10}$$

$$I = \frac{1}{2} \left(\log 10 - \log 5 \right) = \frac{1}{2} \log \left(\frac{10}{5} \right) = \frac{1}{2} \log 2$$

	Old	New
L.L.	2	$2^2+1=5$ (L.L.)
U.L.	3	$3^2+1=10$ (U.L.)

L.L $\rightarrow$ lower limit

U.L $\rightarrow$ upper limit

e.g. $I = \int_0^1 \frac{\tan^{-1}x}{1+x^2} \cdot dx$

$$\text{let } \tan^{-1}x = t$$

$$\Rightarrow \frac{dx}{1+x^2} = dt$$

$$t = \tan^{-1}x \text{ so } x = \tan t$$

	old	New
L.L.	0	$\tan^{-1}(0) = 0$ ✓
U.L.	1	$\tan^{-1}(1) = \frac{\pi}{4}$ ✓

$$I = \int_0^{\frac{\pi}{4}} t \cdot dt = \left(\frac{t^2}{2} \right)_0^{\frac{\pi}{4}} = \left(\frac{\left(\frac{\pi}{4} \right)^2}{2} \right) - \left(\frac{0^2}{2} \right)$$

$$= \frac{\pi^2}{32}$$

Exercise 7.10

Definite Integration by Substitution

$$\int_{x=a}^{x=b} f(x) \cdot dx \rightarrow \int_{\text{L.L. New}}^{\text{U.L. New}} f(t) \cdot dt$$

Q.1 $I = \int_0^1 \frac{x}{x^2+1} \cdot dx$

$t = x^2 + 1$
(U.L. = upper limit
L.L. = lower limit)

let $x^2 + 1 = t$
 $\Rightarrow 2x \cdot dx = dt$
 $\Rightarrow x dx = \frac{dt}{2}$

New U.L. $t = 1^2 + 1 = 2$ ✓
New L.L. $t = 0^2 + 1 = 1$ ✓

$$I = \frac{1}{2} \int_1^2 \frac{dt}{t} = \frac{1}{2} \left[\log|t| \right]_1^2 = \frac{1}{2} \left[\log(2) - \log(1) \right] = \frac{1}{2} \log 2$$

Q.2 $I = \int_0^{\pi/2} \sqrt{\sin \phi} \cdot \cos^5 \phi \cdot d\phi$

$$I = \int_0^{\pi/2} \sqrt{\sin \phi} \cdot \cos^4 \phi \cdot \cos \phi \cdot d\phi$$

$$I = \int_0^{\pi/2} \sqrt{\sin \phi} \cdot (\cos^2 \phi)^2 \cdot \cos \phi \cdot d\phi$$

$$I = \int_0^{\pi/2} \sqrt{\sin \phi} \cdot (1 - \sin^2 \phi)^2 \cdot \cos \phi \cdot d\phi$$

$$I = \int_0^{\pi/2} \sqrt{\sin \phi} \cdot (1 + \sin^4 \phi - 2\sin^2 \phi) \cdot \cos \phi \cdot d\phi$$

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} (1 + \sin^4 \phi - 2\sin^2 \phi) \cdot \cos \phi \, d\phi$$

Let $\sin \phi = t$
 $\Rightarrow \cos \phi \cdot d\phi = dt$

New U.L. $t = \sin \frac{\pi}{2} = 1$ ✓
 New L.L. $t = \sin 0 = 0$ ✓

$$I = \int_0^1 \sqrt{t} (1 + t^4 - 2t^2) \cdot dt$$

$$\sqrt{t} = t^{1/2}$$

$$I = \int_0^1 (t^{1/2} + t^{9/2} - 2t^{5/2}) \cdot dt$$

$$\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + C$$

$$I = \left(\frac{t^{3/2}}{3/2} + \frac{t^{11/2}}{11/2} - 2 \frac{t^{7/2}}{7/2} \right) \Big|_0^1$$

$$I = \left(\frac{2}{3} \cdot 1 + \frac{2}{11} \cdot 1 - \frac{2 \cdot 1 \cdot 2}{7} \right) - (0)$$

$$I = \frac{2}{3} + \frac{2}{11} - \frac{4}{7} = \frac{154 + 42 - 132}{231} = \frac{64}{231} \checkmark$$

Q.3 $I = \int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) \cdot dx$

$$\sin^{-1}(\text{Arg.})$$

$$\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

$$\sin^{-1}(\sin \theta)$$

Put $x = \tan \theta \Rightarrow \tan^{-1} x = \theta$

$$dx = \sec^2 \theta \cdot d\theta$$

New U.L. $= \theta = \tan^{-1}(1) = \frac{\pi}{4}$ ✓

New L.L. $= \theta = \tan^{-1}(0) = 0$ ✓

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \cdot \sec^2\theta \cdot d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta) \cdot \sec^2\theta \cdot d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \underbrace{2\theta}_{\text{I}} \cdot \underbrace{\sec^2\theta}_{\text{II}} \cdot d\theta$$

↓ ↓
I L A T E
① ②

Int. by Parts.

$$\int I \cdot II = I \int II - \int (I' \cdot \int II)$$

$$I = \left[2\theta \cdot \int \sec^2\theta \cdot d\theta - \int \left(\frac{d(2\theta)}{d\theta} \cdot \int \sec^2\theta \cdot d\theta \right) \cdot d\theta \right]_0^{\frac{\pi}{4}}$$

$$I = \left[2\theta \cdot \tan\theta - \int (2 \cdot \tan\theta) \cdot d\theta \right]_0^{\frac{\pi}{4}}$$

$$I = \left[2\theta \tan\theta - 2 \log|\sec\theta| \right]_0^{\frac{\pi}{4}}$$

$$I = \left[2 \cdot \frac{\pi}{4} \cdot \tan \frac{\pi}{4} - 2 \log|\sec \frac{\pi}{4}| \right] - \left[0 - 2 \log|\sec 0| \right]$$

$$I = \frac{\pi}{2} - 2 \log(\sqrt{2}) + 2 \log|1|$$

$$I = \frac{\pi}{2} - \log(\sqrt{2})^2$$

$$I = \frac{\pi}{2} - \log 2$$

✓

Q.4

$$I = \int_0^2 \sqrt{x+2} \cdot dx$$

Hint: put $x+2 = t^2$ Do yourself

$$\begin{aligned} \text{let } x+2 &= t \\ dx &= dt \end{aligned}$$

$$\text{New U.L.} = t = 2+2 = 4$$

$$\text{New L.L.} = t = 0+2 = 2$$

$$\rightarrow I = \int_2^4 (t-2) \cdot \sqrt{t} \cdot dt$$

$$\sqrt{t} = t^{1/2}$$

$$I = \int_2^4 (t^{3/2} - 2t^{1/2}) \cdot dt = \left(\frac{t^{5/2}}{5/2} - 2 \frac{t^{3/2}}{3/2} \right)_2^4$$

$$I = \left(\frac{2t^{5/2}}{5} - \frac{4t^{3/2}}{3} \right)_2^4 = \left(\frac{64}{5} - \frac{32}{3} \right) - \left(\frac{8\sqrt{2}}{5} - \frac{8\sqrt{2}}{3} \right)$$

$$I = 32 \left(\frac{2}{5} - \frac{1}{3} \right) - 8\sqrt{2} \left(\frac{1}{5} - \frac{1}{3} \right)$$

$$I = 32 \left(\frac{6-5}{15} \right) - 8\sqrt{2} \left(\frac{3-5}{15} \right)$$

$$I = \frac{32}{15} + \frac{16\sqrt{2}}{15}$$

$$I = \frac{16 \times 2}{15} + \frac{16\sqrt{2}}{15} = \frac{16\sqrt{2}}{15} (\sqrt{2} + 1)$$

Q.5

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x \cdot dx}{1 + \cos^2 x}$$

Let $\boxed{\cos x = t}$ $\longrightarrow$

New U.L. = $t = \cos \frac{\pi}{2} = 0$

New L.L. = $t = \cos 0 = 1$

$$\Rightarrow -\sin x \cdot dx = dt$$

$$\Rightarrow \sin x \cdot dx = -dt$$

$$I = \int_1^0 \frac{-dt}{1+t^2} = - \left[\tan^{-1} t \right]_1^0$$

$$I = - \left\{ \underbrace{\tan^{-1} 0}_0 - \underbrace{\tan^{-1} 1}_{\frac{\pi}{4}} \right\}$$

$$I = - \left(0 - \frac{\pi}{4} \right) = + \frac{\pi}{4} \quad \checkmark$$

Exercise 7.10 → [Definite Integration by Substitution]
(Q6 to Q10)

$$\int_a^b f(x) \cdot dx \quad \xrightarrow{x \rightarrow t} \quad \int_c^d g(t) \cdot dt$$

Q.6 $I = \int_0^2 \frac{dx}{x+4-x^2}$ (Completing the square method)

$$I = \int_0^2 \frac{dx}{4 - x^2 + x} = \int_0^2 \frac{dx}{4 - (x^2 - x)}$$

$$I = \int_0^2 \frac{dx}{4 - \left[x^2 - x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \right]} = \int_0^2 \frac{dx}{4 + \frac{1}{4} - \left[x - \frac{1}{2} \right]^2}$$

$$I = \int_0^2 \frac{dx}{\frac{17}{4} - \left[x - \frac{1}{2} \right]^2}$$

Substitution,

$$x - \frac{1}{2} = t \quad \checkmark$$

$$dx = dt \quad \checkmark$$

$$I = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2 - t^2}$$

	old (x)	New (t)
U.L.	2	$2 - \frac{1}{2} = \frac{3}{2} \quad \checkmark$
L.L.	0	$0 - \frac{1}{2} = -\frac{1}{2} \quad \checkmark$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a-x}{a+x} \right| + C$$

$$I = \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \cdot \log \left| \frac{\frac{\sqrt{17}}{2} - t}{\frac{\sqrt{17}}{2} + t} \right| \right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$I = \frac{1}{\sqrt{17}} \left[\log \left| \frac{\sqrt{17}-2t}{\sqrt{17}+2t} \right| \right]^{3/2}_{-1/2}$$

$$2t = \cancel{2(3/2)}$$

$$2t = \cancel{2(-1/2)}$$

$$I = \frac{1}{\sqrt{17}} \left\{ \log \left| \frac{\sqrt{17}-3}{\sqrt{17}+3} \right| - \log \left| \frac{\sqrt{17}+1}{\sqrt{17}-1} \right| \right\}$$

$$\log(m) - \log(n) = \log \frac{m}{n}$$

$$I = \frac{1}{\sqrt{17}} \cdot \log \left| \frac{\sqrt{17}-3}{\sqrt{17}+3} \times \frac{\sqrt{17}-1}{\sqrt{17}+1} \right| = \frac{1}{\sqrt{17}} \cdot \log \left| \frac{17-\sqrt{17}-3\sqrt{17}+3}{17+\sqrt{17}+3\sqrt{17}+3} \right|$$

$$I = \frac{1}{\sqrt{17}} \cdot \log \left| \frac{20-4\sqrt{17}}{20+4\sqrt{17}} \right| = \frac{1}{\sqrt{17}} \log \left| \frac{5-\sqrt{17}}{5+\sqrt{17}} \times \frac{5-\sqrt{17}}{5-\sqrt{17}} \right|$$

$$I = \frac{1}{\sqrt{17}} \cdot \log \left| \frac{25+17-10\sqrt{17}}{25-17} \right| = \frac{1}{\sqrt{17}} \cdot \log \left| \frac{42-10\sqrt{17}}{8} \right|$$

$$I = \frac{1}{\sqrt{17}} \log \left| \frac{21-5\sqrt{17}}{4} \right| \quad \checkmark$$

Q. 7 $I = \int_{-1}^1 \frac{dx}{x^2+2x+5}$

$$\left(\frac{1}{Q} \right)$$

Completing the Square method

$$I = \int_{-1}^1 \frac{dx}{\underbrace{x^2+2x+1}_{(a+b)^2} + 4} = \int_{-1}^1 \frac{dx}{(x+1)^2 + 2^2}$$

Substitution, $x+1 = t$
 $dx = dt$

	old	New (+)
U.L.	1	2
L.L.	-1	0

$$I = \int_0^2 \frac{dt}{t^2 + 2^2}$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$a=2$

$$I = \left(\frac{1}{2} \cdot \tan^{-1}\left(\frac{t}{2}\right) \right)_0^2$$

$$I = \frac{1}{2} \left[\tan^{-1}\left(\frac{t}{2}\right) \right]_0^2 = \frac{1}{2} \left[\underbrace{\tan^{-1}(1)}_{\frac{\pi}{4}} - \underbrace{\tan^{-1}(0)}_0 \right]$$

$$I = \frac{1}{2} \left(\frac{\pi}{4} - 0 \right) = \frac{\pi}{8}$$

Q.8 $I = \int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) \cdot e^{2x} \cdot dx$

$$\int e^{ax} (f(x) + f'(x)) \cdot dx = e^{ax} \cdot f(x) + C$$

Let $2x = t$

$x = \frac{t}{2}$

$\Rightarrow 2dx = dt$

$\Rightarrow dx = \frac{dt}{2}$

	Old (x)	New (t)
U.L.	2	4
L.L.	1	2

$$I = \frac{1}{2} \int_2^4 \left(\frac{1}{(t/2)} - \frac{1}{2(t/2)^2} \right) \cdot e^t \cdot dt$$

$$I = \frac{1}{2} \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2} \right) \cdot e^t \cdot dt$$

$$I = \int_2^4 \left(\underbrace{\frac{1}{t}}_{f(t)} - \frac{1}{t^2} \right) \cdot e^t \cdot dt$$

$\uparrow$ $f'(t)$

$$\frac{d\left(\frac{1}{t}\right)}{dt} = -\frac{1}{t^2}$$

$$= \left[e^t \cdot \frac{1}{t} \right]_2^4$$

$$I = \left[e^t \cdot \frac{1}{t} \right]_2^4 = (e^4) \cdot \frac{1}{4} - (e^2) \cdot \frac{1}{2}$$

$$I = e^2 \left(\frac{e^2}{4} - \frac{1}{2} \right) = e^2 \left(\frac{e^2 - 2}{4} \right)$$

[Q.9] The value of the integral $\int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4} \cdot dx$ is $\rightarrow$
 (A) 6 (B) 0
 (C) 3 (D) 4

Ans. $I = \int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4} \cdot dx = \int_{\frac{1}{3}}^1 \frac{(x^3)^{\frac{1}{3}} \left(\frac{1}{x^2} - 1 \right)^{\frac{1}{3}}}{x^4} \cdot dx$

$$I = \int_{\frac{1}{3}}^1 \frac{\left(\frac{1}{x^2} - 1 \right)^{\frac{1}{3}}}{x^3} \cdot dx$$

	Old	New (t)
U.L	1	$\frac{1}{1^2} - 1 = 0$
L.L	$\frac{1}{3}$	8

let $\left(\frac{1}{x^2} - 1 \right) = t$

$$\Rightarrow \frac{-2}{x^3} \cdot dx = dt$$

$$\Rightarrow \frac{1}{x^3} \cdot dx = \frac{dt}{-2}$$

$$I = -\frac{1}{2} \int_8^0 t^{\frac{1}{3}} \cdot \frac{dt}{-2} = -\frac{1}{2} \cdot \left(\frac{t^{\frac{4}{3}}}{\frac{4}{3}} \right)_8^0 = -\frac{3}{8} \left(t^{\frac{4}{3}} \right)_8^0$$

$$I = -\frac{3}{8} \left\{ 0 - 8^{\frac{4}{3}} \right\} = +\frac{3}{8} \cdot (2)^4 = \frac{3}{8} \times 16 = 6$$

$$= 6$$

Q.10 If $f(x) = \int_0^x t \cdot \sin t \cdot dt$, then $f'(x)$ is —

Int. $\rightarrow$ Diff.

$$f(x) = \int_0^x t \cdot \sin t \cdot dt$$

$$I = \int \overset{\text{I}}{t} \cdot \overset{\text{II}}{\sin t} \cdot dt$$

I L A T E

Int by Parts.

$$\boxed{\int I \cdot II = I \int II - \int (I' \cdot II)}$$

$$I = t(-\cos t) - \int (1 \cdot -\cos t) \cdot dt$$

$$I = -t \cos t + \int \cos t \cdot dt$$

$$\boxed{I = -t \cos t + \sin t}$$

$$f(x) = \int_0^x \underbrace{t \cdot \sin t}_{-t \cos t + \sin t} dt = (-t \cos t + \sin t) \Big|_0^x$$

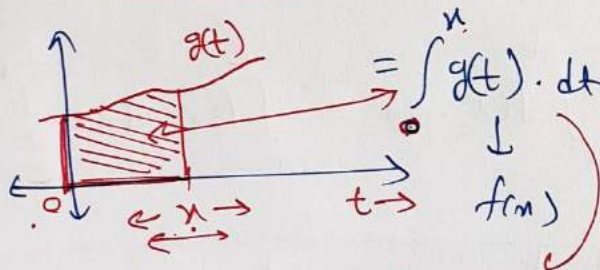
$$f(x) = (-x \cos x + \sin x) - 0$$

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

$$f'(x) = -\cancel{\cos x} + x \sin x + \cancel{\cos x}$$

$$\boxed{f'(x) = x \sin x}$$

Definite Int $\rightarrow$ Area



Area Function

$$\boxed{A(x) = \int_0^x g(t) \cdot dt}$$

$$A'(x) = g(x)$$

$$f(x) = \int_0^x t \sin t \cdot dt$$

$$\boxed{f'(x) = x \sin x}$$

option - B

Some Properties of Definite Integrals

$$(1) \int_a^b f(x) \cdot dx = \int_a^b f(t) \cdot dt$$

$$(2) \int_a^b f(x) \cdot dx = - \int_b^a f(x) \cdot dx$$

$$\int_a^a f(x) \cdot dx = 0$$

$$(3) \int_a^b f(x) \cdot dx = \int_a^c f(x) \cdot dx + \int_c^b f(x) \cdot dx \rightarrow \text{Piece wise} \rightarrow |x|, [x],$$

$$* (4) \int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$$

King Property

$$\int_0^a x(a-x) \quad \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^n x}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}$$

$$\int_0^{\frac{\pi}{2}} \log(\tan x)$$

$$\int_0^{\frac{\pi}{2}} \frac{x \sin x}{\sin^n x + \cos^n x}$$

$$(5) \int_0^a f(x) \cdot dx = \int_0^a f(a-x) \cdot dx$$

$$(6) \int_0^{2a} f(x) \cdot dx = \int_0^a (f(x) + f(2a-x)) \cdot dx$$

→ if $f(2a-x) = f(x)$, then

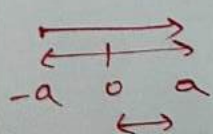
$$\int_0^{2a} f(x) \cdot dx = 2 \int_0^a f(x) \cdot dx$$

→ if $f(2a-x) = -f(x)$, then

$$\int_0^{2a} f(x) \cdot dx = 0$$

$$\int_0^{2\pi} \tan x \quad \int_0^{\pi} \tan x \quad \int_0^{\pi} \log(\tan x)$$

$$(7) \int_{-a}^a f(x) \cdot dx = \int_0^a (f(x) + f(-x)) \cdot dx$$



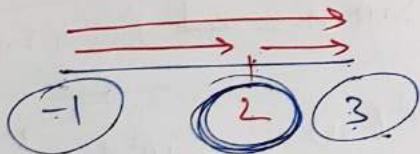
$$\int_{-1}^1 \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

→ if $f(-x) = f(x)$, then $\int_{-a}^a f(x) \cdot dx = 2 \int_0^a f(x) \cdot dx$
Even

→ if $f(-x) = -f(x)$, then $\int_{-a}^a f(x) \cdot dx = 0$
Odd

e.g. $I = \int_{-1}^3 |x-2| \cdot dx$

$|x-2| = \begin{cases} -(x-2), & -\infty < x < 2 \\ +(x-2), & 2 \leq x < \infty \end{cases}$



$|x-2|$
 $x-2=0$
 $x=2$

① $-\infty$ $\downarrow$ 2 $\rightarrow$ ∞ ②

⑥
 $|0-2|$
 $= (-2)$

$I = \int_{-1}^3 |x-2| \cdot dx$

$I = \int_{-1}^2 -(x-2) \cdot dx + \int_2^3 (x-2) \cdot dx$

$I = \int_{-1}^2 (-x+2) \cdot dx + \int_2^3 (x-2) \cdot dx$

$I = \left(-\frac{x^2}{2} + 2x \right)_{-1}^2 + \left(\frac{x^2}{2} - 2x \right)_{2}^3$

$I = (-2+4) - \left(-\frac{1}{2} - 2 \right) + \left(\frac{9}{2} - 6 \right) - (2-4)$

$I = 2 + \left(\frac{5}{2} \right) - \left(\frac{3}{2} \right) + 2$

$I = 5$

e.g. $I_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \cdot dx$

$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cdot dx$

$$\int_{-a}^a f(x) \cdot dx = \int_0^a [f(x) + f(-x)] \cdot dx$$

$I_1, \sin^3 x$

odd

$\sin^3(x) = -\sin^3 x$

$I_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \cdot dx = 0$

$\sin^2 x = \text{Even fn.}$

$\sin^2(-x) = \sin^2 x$

$f(-x) = f(x)$

$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cdot dx$

$I_2 = 2 \int_0^{\frac{\pi}{2}} \sin^2 x \cdot dx$ (1)

King Property

$$\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$$

$I_2 = 2 \int_0^{\frac{\pi}{2}} \sin^2(\frac{\pi}{2} - x) \cdot dx$

$I_2 = 2 \int_0^{\frac{\pi}{2}} \cos^2 x \cdot dx$ (2)

By eqn (1) + (2) $\rightarrow$

$I_2 = \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x) \cdot dx$

$I_2 = \int_0^{\frac{\pi}{2}} 1 \cdot dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0$

$I_2 = \frac{\pi}{2}$

e.g. $\int_0^{\pi/2} \log \sin x \cdot dx \neq I$

$I = \int_0^{\pi/2} \log \sin x \cdot dx \quad \text{--- (1)}$

King Prop.

$$\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$$

$I = \int_0^{\pi/2} \log (\sin(\frac{\pi}{2} - x)) \cdot dx$

$I = \int_0^{\pi/2} \log \cos x \cdot dx \quad \text{--- (2)}$

$(eqn 1) + (eqn 2) \rightarrow 2I = \int_0^{\pi/2} \log \sin x \cdot dx + \int_0^{\pi/2} \log \cos x \cdot dx$

$\Rightarrow 2I = \int_0^{\pi/2} (\log \sin x + \log \cos x) \cdot dx$

$\log m + \log n = \log(mn)$

$\Rightarrow 2I = \int_0^{\pi/2} \log \left(\frac{2 \sin x \cdot \cos x}{2} \right) \cdot dx$

$2 \sin \theta \cdot \cos \theta = \sin 2\theta$

$\Rightarrow 2I = \int_0^{\pi/2} (\log (\sin 2x) - \log 2) \cdot dx$

$\log\left(\frac{m}{n}\right) = \log m - \log n$

$\Rightarrow 2I = \int_0^{\pi/2} \log \sin(2x) \cdot dx$

$-\int_0^{\pi/2} \log 2 \cdot dx$
Constant

Substitution,

Let $2x = t$
 $\Rightarrow 2 dx = dt$
 $\Rightarrow dx = \frac{dt}{2}$

	Old (x)	New (t)
U.L	$\frac{\pi}{2}$	$2(\frac{\pi}{2}) = \pi$
L.L	0	0

$\Rightarrow 2I = \frac{1}{2} \int_0^{\pi} \log \sin t \cdot \underline{dt} - \frac{\pi}{2} \cdot \log 2$

$\int_0^{\pi/2} \log 2 \cdot dx$

$(\log 2) \cdot [x]_0^{\pi/2}$
 $= \log 2 \cdot (\frac{\pi}{2} - 0)$
 $= \frac{\pi}{2} \cdot \log 2$

By using Prop.

$\int_0^a f(x) \cdot dx = \int_0^a (f(x) + f(2a-x)) \cdot dx$

$$\Rightarrow 2I = \frac{1}{2} \int_0^{\pi} \log \sin t \cdot dt - \frac{\pi}{2} \log 2$$

↓ After property

$$\Rightarrow 2I = \frac{1}{2} \int_0^{\frac{\pi}{2}} [\log \sin t + \log \sin(\pi-t)] \cdot dt - \frac{\pi}{2} \log 2$$

$$\Rightarrow 2I = \frac{1}{2} \int_0^{\frac{\pi}{2}} (\log \sin t + \log \sin t) \cdot dt - \frac{\pi}{2} \log 2$$

$$\Rightarrow 2I = \frac{2}{2} \int_0^{\frac{\pi}{2}} (\log \sin t) \cdot dt - \frac{\pi}{2} \log 2$$

P.P.P.

$$\int_a^b f(t) \cdot dt = \int_a^b f(x) \cdot dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log \sin x \cdot dx - \frac{\pi}{2} \log 2$$

Question = I

$$\Rightarrow 2I = I - \frac{\pi}{2} \log 2$$

$$\Rightarrow I = -\frac{\pi}{2} \log 2$$

$$\int_0^{\frac{\pi}{2}} \log \sin x \cdot dx = -\frac{\pi}{2} \log 2$$

$$\int_0^{\frac{\pi}{2}} \log \cos x =$$

Exercise 7.11

Some Properties of Definite Integrals

Q.1 $I = \int_0^{\frac{\pi}{2}} \cos^2 x \cdot dx$ (Evaluate by using the properties) — ①

By King Property $\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right) \cdot dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot dx \quad \text{--- ②}$$

By eqⁿ ① + eqⁿ ② $\rightarrow 2I = \int_0^{\frac{\pi}{2}} \cos^2 x \cdot dx + \int_0^{\frac{\pi}{2}} \sin^2 x \cdot dx$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} (\cos^2 x + \sin^2 x) \cdot dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx = \left[x \right]_0^{\frac{\pi}{2}} = \left(\frac{\pi}{2} - 0 \right)$$

$$\Rightarrow \boxed{I = \frac{\pi}{4}}$$

Q.2 $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \cdot dx$ — ① ✓

by King property.

$$\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} \cdot dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \cdot dx \quad \text{--- ②}$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{(\sqrt{\sin x} + \sqrt{\cos x})}{(\sqrt{\sin x} + \sqrt{\cos x})} \cdot dx$$

by eqn ①
+ eqn ②

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx = (x)_0^{\frac{\pi}{2}} = \left(\frac{\pi}{2} - 0\right)$$

$$\Rightarrow \boxed{I = \frac{\pi}{4}}$$

Q.3 $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} \cdot dx$ — ①

By King Property.

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^{3/2}(\frac{\pi}{2} - x)}{\sin^{3/2}(\frac{\pi}{2} - x) + \cos^{3/2}(\frac{\pi}{2} - x)} \cdot dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^{3/2}(x)}{\cos^{3/2}(x) + \sin^{3/2}(x)} \cdot dx$$
 — ②

By eqn ① + eqn ② →

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cancel{\sin^{3/2} x} + \cancel{\cos^{3/2} x}}{\cancel{\sin^{3/2} x} + \cancel{\cos^{3/2} x}} \cdot dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx = (x)_0^{\frac{\pi}{2}}$$

$$2I = \frac{\pi}{2} - 0 \Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow \boxed{I = \frac{\pi}{4}}$$

Q.4 $I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} \cdot dx$ — (1)

By King Property

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^5(\frac{\pi}{2} - x)}{\sin^5(\frac{\pi}{2} - x) + \cos^5(\frac{\pi}{2} - x)} \cdot dx$$

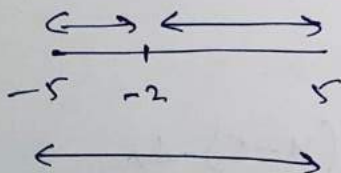
$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} \cdot dx$$
 — (2)

By eqⁿ (1) + eqⁿ (2) $\rightarrow$ $2I = \int_0^{\frac{\pi}{2}} \frac{\cancel{\cos^5 x} + \cancel{\sin^5 x}}{\cancel{\sin^5 x + \cos^5 x}} \cdot dx$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\Rightarrow \boxed{I = \frac{\pi}{4}}$$

Q.5 $I = \int_{-5}^5 |x+2| \cdot dx$

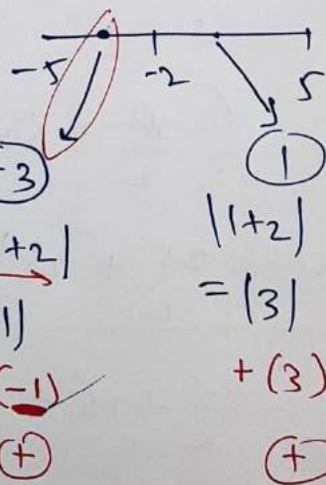


$|x+2|$
 $x+2=0$
 $x=-2$ critical point

$$I = \int_{-5}^{-2} |x+2| \cdot dx + \int_{-2}^5 |x+2| \cdot dx$$

$$I = \int_{-5}^{-2} -(x+2) \cdot dx + \int_{-2}^5 (x+2) \cdot dx$$

$$I = \left(-\frac{x^2}{2} - 2x \right)_{-5}^{-2} + \left(\frac{x^2}{2} + 2x \right)_{-2}^5$$



$$\Rightarrow I = \left[-\frac{x^2}{2} - 2x \right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x \right]_{-2}^5$$

$$\Rightarrow I = \left(-\frac{4}{2} + 4 \right) - \left(-\frac{25}{2} + 10 \right) + \left(\frac{25}{2} + 10 \right) - \left(\frac{4}{2} - 4 \right)$$

$$\Rightarrow I = -2 + 4 + \frac{25}{2} + \frac{25}{2} - 2 + 4$$

$$\Rightarrow I = 25 + 4 = 29 \checkmark$$

Q.6 $I = \int_2^8 |x-5| \cdot dx$

$|x-5|$
 $x-5=0 \Rightarrow x=5$

$$I = \int_2^5 |x-5| \cdot dx + \int_5^8 |x-5| \cdot dx$$

$$I = \int_2^5 -(x-5) \cdot dx + \int_5^8 (x-5) \cdot dx$$

$$I = \left(-\frac{x^2}{2} + 5x \right)_2^5 + \left(\frac{x^2}{2} - 5x \right)_5^8$$

$$I = \left(-\frac{25}{2} + 25 \right) - \left(-\frac{4}{2} + 10 \right) + \left(\frac{64}{2} - 40 \right) - \left(\frac{25}{2} - 25 \right)$$

$$I = -\frac{25}{2} + 25 - \left(-2 + 10 \right) + 32 - 40 - \frac{25}{2} + 25$$

$$I = 34 - 25 = 9 \checkmark$$

Q.7 $I = \int_0^1 x(1-x)^n \cdot dx$

$x \rightarrow 1-x$

By King Prop. $\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$

$I = \int_0^1 (1-x) \cdot (1-(1-x))^n \cdot dx$

$1-(1-x)$
 $= x - x + x$
 $= x$

$I = \int_0^1 (1-x) \cdot (x)^n \cdot dx$

$I = \int_0^1 (x^n - x^{n+1}) \cdot dx = \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right)_0^1$

$I = \left(\frac{1}{n+1} - \frac{1}{n+2} \right) - (0 - 0)$

$I = \frac{x^{n+2} - x^{n+1}}{(n+1) \cdot (n+2)} = \frac{1}{(n+1)(n+2)}$

Q.8 $I = \int_0^{\pi/4} \log(1 + \tan x) \cdot dx$ $\rightarrow \text{log(tan x)}$

By King Prop. $\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$

$I = \int_0^{\pi/4} \log(1 + \tan(\frac{\pi}{4} - x)) \cdot dx$

$I = \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) \cdot dx = \int_0^{\pi/4} \log\left(\frac{1 + \cancel{\tan x} + 1 - \cancel{\tan x}}{1 + \tan x}\right) \cdot dx$

$I = \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right) \cdot dx \rightarrow 2$

By eqⁿ ① + eqⁿ ② $\rightarrow$

$$I + I = \int_0^{\frac{\pi}{4}} \left[\log(1 + \tan x) + \log\left(\frac{2}{1 + \tan x}\right) \right] \cdot dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{4}} \log\left(\cancel{(1 + \tan x)} \cdot \frac{2}{\cancel{(1 + \tan x)}}\right) \cdot dx$$

$\log m + \log n = \log(m \cdot n)$

$$\Rightarrow 2I = \log 2 \int_0^{\frac{\pi}{4}} 1 \cdot dx = \log 2 \cdot (x)_0^{\frac{\pi}{4}}$$

$$\Rightarrow 2I = \log 2 \cdot \left(\frac{\pi}{4} - 0\right)$$

$$\Rightarrow \boxed{I = \frac{\log 2 \cdot \pi}{8}}$$

[Q.9] $I = \int_0^2 (x) \cdot \sqrt{2-x} \cdot dx$

$$\Rightarrow I = \int_0^2 (2-x) \cdot \sqrt{2-(2-x)} \cdot dx$$

$$\Rightarrow I = \int_0^2 (2-x) \cdot \sqrt{x} \cdot dx = \int_0^2 (2 \cdot x^{\frac{1}{2}} - x^{\frac{3}{2}}) \cdot dx$$

$$\Rightarrow I = \left(2 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right)_0^2 = \left(\frac{4}{3} \cdot \frac{3}{2} - \frac{2 \cdot 2^{\frac{5}{2}}}{5} \right) - (0-0)$$

$$\Rightarrow I = \frac{4}{3} (2\sqrt{2}) - \frac{2}{5} (4\sqrt{2}) = \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5} = 8\sqrt{2} \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$I = 8\sqrt{2} \left(\frac{2}{15} \right) = \frac{16\sqrt{2}}{15}$$

King Prop.

$$\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$$

$x \rightarrow (2-x)$ Replace

Exercise 7.11

Some Properties of Definite Integrals

$$\boxed{\text{Q.10}} \quad I = \int_0^{\pi/2} (2 \log \sin u - \log \sin 2u) \cdot du$$

$$I = \int_0^{\pi/2} [\log(\sin^2 u) - \log \sin 2u] \cdot du$$

$$I = \int_0^{\pi/2} \log \left(\frac{\sin^2 u}{\sin 2u} \right) \cdot du$$

$$I = \int_0^{\pi/2} \log \left(\frac{\sin^2 u}{2 \sin u \cos u} \right) \cdot du$$

$$I = \int_0^{\pi/2} \log \left(\frac{\tan u}{2} \right) \cdot du \quad \text{--- (1)}$$

By King Property.

$$\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$$

$$I = \int_0^{\pi/2} \log \left(\frac{\tan(\frac{\pi}{2}-u)}{2} \right) \cdot du$$

$$I = \int_0^{\pi/2} \log \left(\frac{\cot u}{2} \right) \cdot du \quad \text{--- (2)}$$

By eqn (1) + (2) $\rightarrow$

$$2I = \int_0^{\pi/2} \left\{ \log \left(\frac{\tan u}{2} \right) + \log \left(\frac{\cot u}{2} \right) \right\} \cdot du$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log \left(\frac{\tan u}{2} \cdot \frac{\cot u}{2} \right) \cdot du$$

$$\log m^n = n \cdot \log m$$

$$\log m - \log n = \log \frac{m}{n}$$

$$\sin 2u = 2 \sin u \cos u$$

$$\log m + \log n = \log(mn)$$

$$\tan u \cdot \cot u = 1$$

$$2I = \int_0^{\frac{\pi}{2}} \underbrace{\log\left(\frac{1}{4}\right)}_{\text{Constant}} \cdot dx = \log\left(\frac{1}{4}\right) \cdot (x)_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \log\left(\frac{1}{2}\right)^2 \times \left(\frac{\pi}{2} - 0\right)$$

$$\Rightarrow I = \frac{2}{2} \cdot \log\left(\frac{1}{2}\right) \times \left(\frac{\pi}{2}\right) \checkmark$$

[Q.11] $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cdot dx$

Property.

$$\int_{-a}^a f(x) \cdot dx = \int_0^a (f(x) + f(a-x)) \cdot dx$$

$$I = \int_0^{\frac{\pi}{2}} [\sin^2 x + \sin^2(-x)] \cdot dx$$

$\sin^2 x \rightarrow$ even function

$$I = \int_0^{\frac{\pi}{2}} (\sin^2 x + \sin^2 x) \cdot dx$$

$$I = 2 \int_0^{\frac{\pi}{2}} \sin^2 x \cdot dx \quad \text{--- (1)}$$

By King Prop.

$$\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$$

$$I = 2 \int_0^{\frac{\pi}{2}} \sin^2\left(\frac{\pi}{2} - x\right) \cdot dx$$

$$I = 2 \int_0^{\frac{\pi}{2}} \cos^2(x) \cdot dx \quad \text{--- (2)}$$

By eqn (1) + (2) $\therefore 2I = 2 \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x) \cdot dx$ (1)

$$\Rightarrow 2I = 2 \int_0^{\frac{\pi}{2}} 1 \cdot dx \Rightarrow 2I = (x)_0^{\frac{\pi}{2}} \Rightarrow I = \frac{\pi}{2}$$

Q. 12 $\left[I = \int_0^{\pi} \frac{x}{1+\sin x} \cdot dx \right] \text{--- (1)}$

By King Prop.

$$\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi-x}{1+\sin(\pi-x)} \cdot dx$$

$$x \rightarrow \pi-x$$

$$\sin(180^\circ - \theta) = \sin \theta$$

$$\Rightarrow \left[I = \int_0^{\pi} \frac{\pi-x}{1+\sin x} \cdot dx \right] \text{--- (2)}$$

By eqn (1) + (2) $\rightarrow 2I = \int_0^{\pi} \frac{x + (\pi-x)}{(1+\sin x)} \cdot dx$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1}{1+\sin x} \cdot dx$$

$$\Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \frac{1}{1+\sin x} \times \frac{(1-\sin x)}{(1-\sin x)} \cdot dx$$

$$\Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \frac{(1-\sin x)}{\cos^2 x} \cdot dx$$

$$\begin{aligned} (1+\sin x) \cdot (1-\sin x) \\ = 1 - \sin^2 x \\ = \cos^2 x \end{aligned}$$

$$\Rightarrow I = \frac{\pi}{2} \int_0^{\pi} (\sec^2 x - \sec x \cdot \tan x) \cdot dx$$

$$\Rightarrow I = \frac{\pi}{2} \cdot (\tan x - \sec x) \Big|_0^{\pi}$$

$$\Rightarrow I = \frac{\pi}{2} \left\{ (\tan \pi - \sec \pi) - (\tan 0 - \sec 0) \right\}$$

$$\Rightarrow I = \frac{\pi}{2} \{ 1 + 1 \} = \pi$$

Q.13 $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \cdot dx$

Prop. $\int_{-a}^a f(x) \cdot dx = \int_0^a [f(x) + f(-x)] \cdot dx$

$$I = \int_0^{\frac{\pi}{2}} (\sin^7 x + \sin^7(-x)) \cdot dx$$

$$I = \int_0^{\frac{\pi}{2}} (\cancel{\sin^7 x} - \cancel{\sin^7 x}) \cdot dx$$

$$I = \int_0^{\frac{\pi}{2}} 0 \cdot dx = 0$$

$$\begin{aligned} [\sin(-x)]^7 \\ = -(\sin x)^7 \end{aligned}$$

Q.14 $I = \int_0^{2\pi} \cos^5 x \cdot dx$

Prop. $\int_0^{2a} f(x) \cdot dx = \int_0^a (f(x) + f(2a-x)) \cdot dx$

$$I = \int_0^{\pi} (\cos^5 x + \cos^5(2\pi-x)) \cdot dx$$

$$I = \int_0^{\pi} (\cos^5 x + \cos^5 x) \cdot dx$$

$$I = 2 \int_0^{\pi} \cos^5 x \cdot dx$$

Same Prop. (again)

$$I = 2 \int_0^{\frac{\pi}{2}} (\cos^5 x + \cos^5(\pi-x)) \cdot dx$$

$$I = 2 \int_0^{\frac{\pi}{2}} (\cancel{\cos^5 x} + \cancel{(-\cos x)^5}) \cdot dx = 0$$

$$\begin{aligned} \cos(2\pi-\theta) \\ = \cos \theta \end{aligned}$$

$$\pi \rightarrow \cancel{2\pi} \text{ (iv)}$$

$$\begin{aligned} \cos(\pi-\theta) \\ = -\cos \theta \end{aligned}$$

Q.15
$$I = \int_0^{\pi/2} \left(\frac{\sin x - \cos x}{1 + \sin x \cdot \cos x} \right) \cdot dx \quad \text{--- (1)}$$

By King Prop.
$$\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$$

$x \rightarrow \frac{\pi}{2} - x$

$$I = \int_0^{\pi/2} \frac{\sin(\frac{\pi}{2} - x) - \cos(\frac{\pi}{2} - x)}{1 + \sin(\frac{\pi}{2} - x) \cdot \cos(\frac{\pi}{2} - x)} \cdot dx$$

$$I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \cdot \sin x} \cdot dx \quad \text{--- (2)}$$

By eqⁿ (1) + (2) : $\rightarrow$

$$2I = \int_0^{\pi/2} \frac{(\cancel{\sin x - \cos x}) + (\cancel{\cos x - \sin x})}{(1 + \sin x \cdot \cos x)} \cdot dx$$

$$I = 0 \quad \checkmark$$

~~Q.16~~

★ Q.16 $I = \int_0^{\pi} \log(1 + \cos x) \cdot dx$

Prop. $\int_0^{2a} f(x) \cdot dx = \int_0^a (f(x) + f(2a-x)) \cdot dx$

$$I = \int_0^{\frac{\pi}{2}} [\log(1 + \cos x) + \log(1 + \cos(\pi - x))] \cdot dx$$

$\cos(\pi - x) = -\cos x$

$$I = \int_0^{\frac{\pi}{2}} [\log(1 + \cos x) + \log(1 - \cos x)] \cdot dx$$

$\log m + \log n = \log mn$

$$I = \int_0^{\frac{\pi}{2}} \log(\sin^2 x) \cdot dx$$

$(1 + \cos \theta)(1 - \cos \theta) = \sin^2 \theta$

$$I = 2 \int_0^{\frac{\pi}{2}} \log(\sin x) \cdot dx$$

$$\Rightarrow \frac{I}{2} = \int_0^{\frac{\pi}{2}} \log(\sin x) \cdot dx$$

By King prop.

① $\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$

$$\Rightarrow \frac{I}{2} = \int_0^{\frac{\pi}{2}} \log(\sin(\frac{\pi}{2} - x)) \cdot dx$$

$$\Rightarrow \frac{I}{2} = \int_0^{\frac{\pi}{2}} \log(\cos x) \cdot dx$$

By eqⁿ ① + eqⁿ ② $\rightarrow I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) \cdot dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{2 \sin x \cdot \cos x}{2}\right) \cdot dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left[\log(\sin 2x) - \log 2 \right] \cdot dx$$

$$\Rightarrow I = \int_0^{\pi/2} \log(\sin 2x) \cdot dx - \int_0^{\pi/2} \log 2 \cdot dx$$

Substitution,

$$\boxed{2x = t}$$

$$\Rightarrow 2dx = dt$$

$$\Rightarrow dx = \frac{dt}{2} \checkmark$$

	(t) New
U.L	π
L.L	0

$$I = \frac{1}{2} \int_0^{\pi} \log(\sin t) \cdot dt - \frac{\pi}{2} \log 2$$

Prop. $\int_0^{2a} f(x) \cdot dx = \int_0^a (f(x) + f(2a-x)) \cdot dx$

$$\begin{aligned} & \log 2 \cdot \int_0^{\pi/2} 1 \cdot dx \\ &= \log 2 \cdot (x)_0^{\pi/2} \\ &= \log 2 \cdot \left(\frac{\pi}{2} - 0\right) \\ &= \frac{\pi}{2} \cdot \log 2 \end{aligned}$$

$$I = \frac{1}{2} \int_0^{\pi/2} \left(\log \sin t + \log \sin(\pi - t) \right) \cdot dt - \frac{\pi}{2} \log 2$$

$$I = \frac{1}{2} \int_0^{\pi/2} 2 \cdot \log \sin t \cdot dt - \frac{\pi}{2} \log 2$$

$$\int_a^b f(x) \cdot dx = \int_a^b f(t) \cdot dt$$

$$\Rightarrow I = \int_0^{\pi/2} \log \sin x \cdot dx - \frac{\pi}{2} \log 2$$

by eqn (1) $\rightarrow I/2$

$$\Rightarrow I = I/2 - \frac{\pi}{2} \log 2$$

$$\Rightarrow \left(\frac{I}{2}\right) = -\frac{\pi}{2} \log 2 \rightarrow \boxed{I = -\pi \log 2}$$

Exercise 7.11

Some Properties of Definite Integrals

(Q17 to Q24)

Q.17 $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} \cdot dx$ By King Prop.

$x \rightarrow a-x$

$\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$

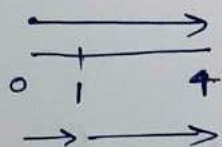
$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x-(a-x)}} \cdot dx$

$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} \cdot dx$ — (2)

By eqn (1) + (2) $\rightarrow 2I = \int_0^a \frac{(\sqrt{x} + \sqrt{a-x})}{(\sqrt{x} + \sqrt{a-x})} \cdot dx$

$\Rightarrow 2I = \int_0^a 1 \cdot dx = (x)_0^a = (a-0) = a \Rightarrow I = \frac{a}{2}$

Q.18 $I = \int_0^4 |x-1| \cdot dx$



$|x-1|$
0

$x-1=0$
 $x=1 \leftarrow$ Critical Point

$I = \int_0^1 |x-1| \cdot dx + \int_1^4 |x-1| \cdot dx$

$| - | = -(-) \rightarrow \oplus$
 $| + | = +(+) = \oplus$

$I = \int_0^1 -(x-1) \cdot dx + \int_1^4 +(x-1) \cdot dx$

$I = \left(-\frac{x^2}{2} + x\right)_0^1 + \left(\frac{x^2}{2} - x\right)_1^4$

$$I = \left(-\frac{x^2}{2} + x\right)_0^1 + \left(\frac{x^2}{2} - x\right)_1^4$$

$$I = \underbrace{\left(-\frac{1}{2} + 1\right) - (0)} + \underbrace{\left(\frac{16}{2} - 4\right) - \left(\frac{1}{2} - 1\right)}$$

$$I = \cancel{\left(-\frac{1}{2}\right)} + 1 + 8 - 4 - \cancel{\left(\frac{1}{2}\right)} + 1 = 5 \quad \checkmark$$

Q.19 Show that $\int_0^a f(x) \cdot g(x) \cdot dx$

Q.19 Show that $\int_0^a f(x) \cdot g(x) \cdot dx = 2 \int_0^a f(x) \cdot dx$, if 'f' & 'g' are defined as $\underbrace{f(x) = f(a-x)}_{\text{Given}}$ and $\underbrace{g(x) + g(a-x) = 4}_{\text{Given}}$.

Ans. LHS = $I = \int_0^a f(x) \cdot g(x) \cdot dx$ — (1)

$x \rightarrow 0+a-x$ By King Prop.

$$\left(\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx \right)$$

$$I = \int_0^a \underbrace{f(a-x)}_{f(x)} \cdot \underbrace{g(a-x)}_{4-g(x)} \cdot dx \quad (\text{given})$$

$$I = \int_0^a f(x) \cdot (4 - g(x)) \cdot dx = \int_0^a [4 \cdot f(x) - f(x) \cdot g(x)] \cdot dx$$

$$I = 4 \int_0^a f(x) \cdot dx - \underbrace{\int_0^a f(x) \cdot g(x) \cdot dx}_I \quad (\text{by eqn (1)})$$

$$I = 4 \int_0^a f(x) \cdot dx - I$$

$$\Rightarrow 2I = 4 \int_0^a f(x) \cdot dx \Rightarrow \boxed{I = 2 \int_0^a f(x) \cdot dx} = \text{RHS.}$$

Q.20 The value of $\int_{-\pi/2}^{\pi/2} (x^3 + x \cos x + \tan^5 x + 1) \cdot dx$

(A) 0 (B) 2 (C) ~~π~~ (D) 1

Ans.

$$I = \int_{-\pi/2}^{\pi/2} \underbrace{(x^3 + x \cos x + \tan^5 x + 1)}_{f(x)} \cdot dx$$

By prop.

$$\int_{-a}^a f(x) \cdot dx = \int_0^a [f(x) + f(-x)] \cdot dx$$

$$\begin{aligned} \cos(-\theta) \\ = \cos \theta \end{aligned}$$

$$I = \int_0^{\pi/2} \left[(x^3 + x \cos x + \tan^5 x + 1) + ((-x)^3 + (-x) \cdot \cos(-x) + \tan^5(-x) + 1) \right] \cdot dx$$

$$I = \int_0^{\pi/2} \left[\cancel{x^3} + \cancel{x \cos x} + \cancel{\tan^5 x} + 1 + (-\cancel{x^3} - \cancel{x \cos x} - \cancel{\tan^5 x} + 1) \right] \cdot dx$$

(1+1=2)

$$I = \int_0^{\pi/2} (2) \cdot dx = (2x) \Big|_0^{\pi/2} = 2 \times \frac{\pi}{2} - 2 \times 0 = \pi$$

$$\underline{I = \pi} \checkmark$$

[Q.21] The value of $\int_0^{\pi/2} \log \left(\frac{4+3\sin x}{4+3\cos x} \right) \cdot dx$ is -

- (A) 2 (B) $\frac{3}{4}$ (C) 0 (D) -2

Ans. $I = \int_0^{\pi/2} \log \left(\frac{4+3\sin x}{4+3\cos x} \right) \cdot dx$ — (1)

By King Prop $\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$
 $x \rightarrow 0 + \frac{\pi}{2} - x$

$$I = \int_0^{\pi/2} \log \left(\frac{4+3\sin(\frac{\pi}{2}-x)}{4+3\cos(\frac{\pi}{2}-x)} \right) \cdot dx$$

$$I = \int_0^{\pi/2} \log \left(\frac{4+3\cos x}{4+3\sin x} \right) \cdot dx$$
 — (2)

By eqn (1) + (2) $\rightarrow$

$$2I = \int_0^{\pi/2} \left[\log \left(\frac{4+3\sin x}{4+3\cos x} \right) + \log \left(\frac{4+3\cos x}{4+3\sin x} \right) \right] \cdot dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log \left(\frac{4+3\sin x}{4+3\cos x} \times \frac{4+3\cos x}{4+3\sin x} \right) \cdot dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log(1) \cdot dx$$

$\log 1 = 0$

$$\Rightarrow 2I = \int_0^{\pi/2} 0 \cdot dx$$

$$\Rightarrow 2I = 0 \Rightarrow I = 0$$

Miscellaneous Exercise on Chapter 7

Q.1 Integrate the function

$$I = \int \frac{1}{x-x^3} \cdot dx$$

(I)

$$\frac{1}{x-x^3} = \frac{1}{x(1-x^2)} = \frac{1}{x(1-x)(1+x)}$$

↓

$$\frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$

✓

(II)

$$I = \int \frac{1}{x-x^3} \cdot dx$$

$$I = \int \frac{1}{x^3 \left(\frac{1}{x^2} - 1 \right)} \cdot dx$$

↓
t

By substitution

$$\left(\frac{1}{x^2} - 1 \right) = t \Rightarrow \left(\frac{-2}{x^3} \right) \cdot dx = dt$$

$$\Rightarrow \frac{1}{x^3} \cdot dx = \frac{dt}{-2}$$

$$I = \frac{1}{-2} \int \frac{dt}{t} = -\frac{1}{2} \log |t| + C$$

$$I = -\frac{1}{2} \cdot \log \left| \frac{1}{x^2} - 1 \right| + C = -\frac{1}{2} \log \left| \frac{1-x^2}{x^2} \right| + C$$

$$I = \frac{1}{2} \log \left| \frac{x^2}{1-x^2} \right| + C$$

Q.2 $I = \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} \cdot dx$

Rationalisation

$$I = \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} \times \frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+a} - \sqrt{x+b}} \cdot dx$$

$$I = \int \frac{\sqrt{x+a} - \sqrt{x+b}}{(x+a) - (x+b)} \cdot dx = \int \frac{\sqrt{x+a} - \sqrt{x+b}}{a-b} \cdot dx$$

$$I = \frac{1}{a-b} \int ((x+a)^{\frac{1}{2}} - (x+b)^{\frac{1}{2}}) \cdot dx$$

$$I = \frac{1}{(a-b)} \cdot \left\{ \frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}} \right\} + C$$

$\frac{1}{2} + 1 = \frac{3}{2}$

$\int x^n = \frac{x^{n+1}}{(n+1)}$

$$I = \frac{2}{3(a-b)} \cdot ((x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}) + C$$

Q.3 $I = \int \frac{1}{x \sqrt{ax-x^2}} \cdot dx$ [Hint: put $x = \frac{a}{t}$]

$$I = \int \frac{1}{x \sqrt{x^2 \left(\frac{a}{x} - 1 \right)}} \cdot dx$$

x^2 (common)

$$I = \int \frac{1}{x^2 \sqrt{\frac{a}{x} - 1}} \cdot dx$$

$\frac{a}{x} - 1 \rightarrow t$

By Substitution

$$\boxed{\frac{a}{x} - 1 = t} \checkmark$$

$$-\frac{a}{x^2} \cdot dx = dt$$

$$\boxed{\frac{1}{x^2} \cdot dx = \frac{dt}{-a}} \checkmark$$

$$I = -\frac{1}{a} \int \frac{dt}{\sqrt{t}}$$

$$I = -\frac{1}{a} \int t^{-\frac{1}{2}} \cdot dt$$

$$I = -\frac{1}{a} \left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}} \right) + C = -\frac{2}{a} \sqrt{\frac{a}{x} - 1} + C$$

$$\boxed{I = -\frac{2}{a} \sqrt{\frac{a-x}{x}} + C}$$

Q.4 $I = \int \frac{1}{x^2 (x^4 + 1)^{3/4}} \cdot dx$

x^4 common

Put $x = \frac{1}{t}$

$$I = \int \frac{1}{x^2 \left[x^4 \left(1 + \frac{1}{x^4} \right) \right]^{3/4}} \cdot dx = \int \frac{1}{x^5 \left(1 + \frac{1}{x^4} \right)^{3/4}} \cdot dx$$

t

By Substitution,

$1 + \frac{1}{x^4} = t$

$\Rightarrow -\frac{4}{x^5} \cdot dx = dt \Rightarrow \frac{1}{x^5} \cdot dx = \frac{dt}{-4}$

$$I = \frac{1}{-4} \int \frac{dt}{t^{3/4}} = -\frac{1}{4} \int t^{-3/4} \cdot dt$$

x^n

$$I = -\frac{1}{4} \cdot \frac{(t^{1/4})}{(1/4)} + C = -\left(1 + \frac{1}{x^4}\right)^{1/4} + C$$

Q.5 $I = \int \frac{1}{x^{1/2} + x^{1/3}} \cdot dx$

Hint:

$\frac{1}{x^{1/2} + x^{1/3}} = \frac{1}{x^{1/3} (1 + x^{1/6})}$

Put $x = t^6$

Pattern $\frac{1}{x^m + x^n + x^p}$

Put $x = t^{\text{LCM}(m, n, p)}$

$\text{LCM}(2, 3) = 6$

Put $x = (t^6) \rightarrow dx = 6 \cdot t^5 \cdot dt$

$$I = \int \frac{1}{(t^6)^{1/2} + (t^6)^{1/3}} \cdot 6 \cdot t^5 \cdot dt = 6 \int \frac{t^5 \cdot dt}{t^3 + t^2}$$

$$I = 6 \int \frac{t^5}{t^3 + t^2} \cdot dt = 6 \int \frac{t^3}{t^2(t+1)} \cdot dt$$

$$I = 6 \int \frac{t^3}{(t+1)} \cdot dt$$

$$\begin{array}{r} t+1 \overline{) t^3} \quad (t^2 - t + 1) \\ \underline{-(t^2 + t^2)} \\ -t^2 \end{array}$$

$$I = 6 \int \left((t^2 - t + 1) - \frac{1}{t+1} \right) \cdot dt$$

$$I = 6 \left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1| \right] + C$$

$$\begin{array}{r} -t^2 \\ -t^2 - t \\ \underline{+t} \\ t \\ t+1 \\ \underline{-} \\ -1 \end{array}$$

$$x = t^6 \Rightarrow x^{\frac{1}{6}} = t$$

$$I = 6 \left(\frac{(x^{\frac{1}{6}})^3}{3} - \frac{(x^{\frac{1}{6}})^2}{2} + x^{\frac{1}{6}} - \log|x^{\frac{1}{6}}+1| \right) + C$$

$$I = 2 \cdot x^{\frac{1}{2}} - 3 \cdot x^{\frac{1}{3}} + 6 \cdot x^{\frac{1}{6}} - 6 \cdot \log|x^{\frac{1}{6}}+1| + C$$

Q.6 $I = \int \frac{5x}{(x+1) \cdot (x^2+9)} \cdot dx$

Partial Fraction.

$$\frac{5x}{(x+1) \cdot (x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

$$\Rightarrow 5x = A(x^2+9) + (Bx+C) \cdot (x+1)$$

$$\Rightarrow 5x = \underline{Ax^2} + 9A + \underline{Bx^2} + \underline{Bx} + \underline{Cx} + C$$

Comparison

$$x^2 \quad 0 = A+B \quad \text{--- (1)}$$

$$x \quad 5 = B+C \quad \text{--- (2)}$$

Constant
 $0 = 9A+C \quad \text{--- (3)}$

Q.6 $I = \int \frac{5x}{(x+1)(x^2+9)} \cdot dx$

Partial Fraction

$$\frac{5x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

$$\Rightarrow 5x = A(x^2+9) + (Bx+C)(x+1)$$

$$\Rightarrow 5x = \underbrace{Ax^2 + 9A}_{\uparrow} + \underbrace{Bx^2 + Bx + Cx + C}_{\uparrow \quad \uparrow \quad \uparrow}$$

By Comparison

x^2

$$0 = A + B \quad \text{--- (1)}$$

x

$$5 = B + C \quad \text{--- (2)}$$

Constant

$$0 = 9A + C \quad \text{--- (3)}$$

By Solving eqⁿ (1), (2) & (3) $\rightarrow$

$$A = -\frac{1}{2}$$

$$B = \frac{1}{2}$$

$$C = \frac{9}{2}$$

$$I = \int \frac{5x}{(x+1)(x^2+9)} \cdot dx$$

$$I = \int \left(\frac{(-\frac{1}{2})}{x+1} + \frac{(\frac{1}{2})x + (\frac{9}{2})}{x^2+9} \right) \cdot dx$$

$$I = -\frac{1}{2} \int \frac{1}{x+1} \cdot dx + \frac{1}{2} \int \frac{2x}{x^2+9} \cdot dx + \frac{9}{2} \int \frac{1}{x^2+9} \cdot dx$$

$\xrightarrow{dt}$
 $\xrightarrow{t}$

$$I = -\frac{1}{2} \log|x+1| + \frac{1}{4} \int \frac{dt}{t} + \frac{9}{2} \left(\frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) \right) + C$$

$$I = -\frac{1}{2} \log|x+1| + \frac{1}{4} \log|t| + \frac{3}{2} \tan^{-1}\left(\frac{x}{3}\right) + C$$

$$I = -\frac{1}{2} \log|x+1| + \frac{1}{4} \log|x^2+9| + \frac{3}{2} \tan^{-1}\left(\frac{x}{3}\right) + C$$

$$\boxed{Q.7} \quad I = \int \frac{\sin x}{\sin(x-a)} \cdot \underline{dx}$$

7.2 Example

Let $x-a = t$
 $\Rightarrow dx = \underline{dt}$

$$I = \int \frac{\sin(t+a)}{\sin(t)} \cdot dt$$

$$I = \int \frac{\sin t \cdot \cos a + \sin a \cdot \cos t}{\sin t} \cdot dt$$

$a \rightarrow \text{constant}$
 $t \rightarrow \text{variable}$

$$I = \int (\cos a + \sin a \cdot \cot t) \cdot dt$$

$$I = \cos a \int 1 \cdot dt + \sin a \int \cot t \cdot dt$$

$$I = \cos a \cdot \underline{\left(\frac{t}{1}\right)} + \sin a \cdot \log |\sin t| + C$$

$$I = \underline{(x-a) \cdot \cos a} + \underline{\sin a \cdot \log |\sin(x-a)|} + C$$

$$I = \underline{x \cos a} - \underline{a \cos a} + \underline{\sin a \cdot \log |\sin(x-a)|} + \underline{C}$$

Constant

Constant

C_1

New Constant

$$I = x \cos a + \sin a \cdot \log |\sin(x-a)| + C_1$$

Q.8

$$I = \int \frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}} \cdot dx$$

$$\log m^n = n \cdot \log m$$

Maths

$$\log x = \log_e x = \log x$$

2.718...

$\log_a x = x$
↑
same

$$I = \int \frac{e^{\log_e(x^5)} - e^{\log_e x^4}}{e^{\log_e x^3} - e^{\log_e x^2}} \cdot dx = \int \frac{x^5 - x^4}{x^3 - x^2} \cdot dx$$

$$I = \int \frac{x^4(x-1)}{x^2(x-1)} \cdot dx = \int x^2 \cdot dx = \frac{x^3}{3} + C$$

Q.9

$$I = \int \frac{\cos x}{\sqrt{4 - \sin^2 x}} \cdot dx$$

$\sin x = t$ (let) —

$\cos x \cdot dx = dt$ —

$$I = \int \frac{dt}{\sqrt{4 - t^2}} = \int \frac{dt}{\sqrt{2^2 - t^2}} = \sin^{-1} \left(\frac{t}{a} \right) + C$$

$$= \sin^{-1} \left(\frac{\sin x}{2} \right) + C$$

$a = 2$



Miscellaneous Exercise on Chapter 7

Q.10 $I = \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} \cdot dx$

$$I = \int \frac{(\sin^4 x - \cos^4 x) \cdot (\sin^4 x + \cos^4 x)}{1 - 2\sin^2 x \cos^2 x} \cdot dx$$

$$\underline{a^2 - b^2 = (a+b)(a-b)}$$

$$\sin^8 x - \cos^8 x = (\sin^4 x)^2 - (\cos^4 x)^2$$

Class-10 Q.10

$$\begin{matrix} a = \sin^2 x \\ b = \cos^2 x \end{matrix}$$

$$\underline{a^2 + b^2 = (a+b)^2 - 2ab}$$

$$I = \int \frac{(\sin^2 x + \cos^2 x) (\sin^2 x - \cos^2 x) \cdot \left[(\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x \right]}{1 - 2\sin^2 x \cos^2 x} \cdot dx$$

$\textcircled{1}^2 = 1$

$$I = \int \frac{(\sin^2 x - \cos^2 x) \cdot (1 - 2\sin^2 x \cos^2 x)}{(1 - 2\sin^2 x \cos^2 x)} \cdot dx$$

$$I = - \int (\cos^2 x - \sin^2 x) \cdot dx = - \int \cos 2x \cdot dx$$

$$\boxed{I = - \frac{\sin 2x}{2} + C}$$



Q11 $I = \int \frac{1}{\cos(x+a) \cdot \cos(x+b)} \cdot dx$

multiply and Divide $\sin(a-b)$

$$\begin{array}{l} (x+a) \checkmark \\ -(x+b) \checkmark \\ \hline (a-b) \end{array}$$

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x+a) \cdot \cos(x+b)} \cdot dx$$

$$\rightarrow \sin(A-B)$$

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin((x+a) - (x+b))}{\cos(x+a) \cdot \cos(x+b)} \cdot dx$$

$$I = \frac{1}{\sin(a-b)} \cdot \int \frac{\sin(x+a) \cdot \cos(x+b) - \cos(x+a) \cdot \sin(x+b)}{\cos(x+a) \cdot \cos(x+b)} \cdot dx$$

$$I = \frac{1}{\sin(a-b)} \cdot \int [\tan(x+a) - \tan(x+b)] \cdot dx$$

$$I = \frac{1}{\sin(a-b)} \cdot \left[\log |\sec(x+a)| - \log |\sec(x+b)| \right] + C$$

$\log m - \log n = \log \left(\frac{m}{n} \right)$

$$I = \frac{1}{\sin(a-b)} \cdot \log \left| \frac{\sec(x+a)}{\sec(x+b)} \right| + C$$

$$I = \frac{1}{\sin(a-b)} \cdot \log \left| \frac{\cos(x+b)}{\cos(x+a)} \right| + C$$

$$\sec \theta = \frac{1}{\cos \theta}$$

Q.12

$$I = \int \frac{x^3}{\sqrt{1-x^8}} \cdot dx$$

$$x^8 = (x^4)^2$$

$$I = \int \frac{x^3 \cdot dx}{\sqrt{1-(x^4)^2}}$$

$$x^4 = t$$

$$4x^3 \cdot dx = dt$$

$$x^3 \cdot dx = \frac{dt}{4}$$

$$I = \frac{1}{4} \int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{4} \sin^{-1}\left(\frac{t}{1}\right) + C$$

$$= \frac{1}{4} \sin^{-1}(x^4) + C$$

Q.13

$$I = \int \frac{e^x}{(1+e^x)(2+e^x)} \cdot dx$$

$$e^x = t$$

$$e^x \cdot dx = dt$$

$$I = \int \frac{dt}{(1+t)(2+t)}$$

Partial fraction

$$\frac{1}{(1+t)(2+t)} = \frac{A}{(1+t)} + \frac{B}{(2+t)} \Rightarrow 1 = A(2+t) + B(1+t)$$
$$1 = 2A + At + B + Bt$$

Comparison, $t \rightarrow 0 = A + B$

Constant $\rightarrow 1 = 2A + B$

$$B = -1$$

$$1 = 2A \Rightarrow A = \frac{1}{2}$$

$$I = \int \frac{dt}{(1+t)(2+t)} = \int \left(\frac{1}{1+t} - \frac{1}{2+t} \right) \cdot dt$$

$$I = \log|1+t| - \log|2+t| + C = \log\left|\frac{1+t}{2+t}\right| + C = \log\left|\frac{1+e^x}{2+e^x}\right| + C$$

Q.14 $I = \int \frac{1}{(x^2+1)(x^2+4)} \cdot dx$

Partial Fraction

$\frac{1}{(x^2+1)(x^2+4)} \rightarrow \text{let } x^2 = y$

$\frac{1}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4} = \frac{\left(\frac{1}{3}\right)}{y+1} + \frac{\left(-\frac{1}{3}\right)}{y+4}$

$\begin{matrix} \xrightarrow{y=0} \\ x^2 = -1 \end{matrix}$ $\begin{matrix} \xrightarrow{y=0} \\ x^2 = -4 \end{matrix}$

$\frac{1}{(x^2+1)(x^2+4)} = \frac{\frac{1}{3}}{(x^2+1)} + \frac{-\frac{1}{3}}{x^2+4}$

$I = \int \frac{1}{(x^2+1)(x^2+4)} \cdot dx = \frac{1}{3} \int \frac{1}{x^2+1} \cdot dx - \frac{1}{3} \int \frac{1}{x^2+4} \cdot dx$

$I = \frac{1}{3} \left[\frac{1}{1} \tan^{-1}\left(\frac{y}{1}\right) \right] - \frac{1}{3} \left[\frac{1}{2} \tan^{-1}\left(\frac{y}{2}\right) \right] + C$

$I = \frac{1}{3} \tan^{-1}(x) - \frac{1}{6} \tan^{-1}\left(\frac{x}{2}\right) + C$

Q.15 $I = \int \cos^3 x \cdot e^{\log \sin x} \cdot dx$

matrix
 $\log x = \log_e x$

$I = \int \cos^3 x \cdot \sin x \cdot dx$

$\log x = x$

let $\cos x = t$

$\Rightarrow -\sin x \cdot dx = dt$

$I = -\frac{t^4}{4} + C$

$I = -\int t^3 \cdot dt \Rightarrow I = -\frac{\cos^4 x}{4} + C$

Q.16 $I = \int e^{3 \log x} \cdot (x^4 + 1)^{-1} \cdot dx$

$n \cdot \log m = \log (m^n)$

$$I = \int e^{\log_e x^3} \cdot \frac{1}{(x^4 + 1)} \cdot dx = \int x^3 \cdot \frac{1}{(x^4 + 1)} \cdot dx$$

Let $x^4 + 1 = t$
 $\Rightarrow 4x^3 \cdot dx = dt$
 $\Rightarrow x^3 \cdot dx = \frac{dt}{4}$

$$I = \frac{1}{4} \int \frac{dt}{t} = \frac{1}{4} \log |t| + C$$

$$\Rightarrow I = \frac{1}{4} \log |x^4 + 1| + C$$

Q.17

$$I = \int \underline{f'(ax+b)} \cdot \left[\underline{f(ax+b)} \right]^n \cdot \underline{dx}$$

Let $f(ax+b) = t$

$$\Rightarrow \underline{f'(ax+b)} \cdot (a) \cdot \underline{dx} = dt$$

$$\Rightarrow \underline{f'(ax+b)} \cdot dx = \underline{\frac{dt}{a}}$$

$$I = \int t^n \cdot \frac{dt}{a}$$

$$I = \frac{1}{a} \left(\frac{t^{n+1}}{n+1} \right) + C$$

$$I = \frac{1}{a(n+1)} \cdot \left[f(ax+b) \right]^{n+1} + C$$

Miscellaneous Exercise on Chapter (7)

$$\boxed{\text{Q.18}} \quad I = \int \frac{1}{\sqrt{\sin^3 x \cdot \sin(x+a)}} \cdot dx$$

$$I = \int \frac{1}{\sqrt{\sin^3 x \cdot [\sin x \cdot \cos a + \cos x \cdot \sin a]}} \cdot dx$$

$$I = \int \frac{1}{\sqrt{\sin^4 x \left\{ \cos a + \frac{\cos x \cdot \sin a}{\sin x} \right\}}} \cdot dx$$

$$I = \int \frac{1}{\sin^2 x \sqrt{\cos a + \cot x \cdot \sin a}} \cdot dx$$

$$I = \int \frac{\operatorname{cosec}^2 x \cdot dx}{\sqrt{\cos a + \cot x \cdot \sin a}}$$

↓

$$I = -\frac{1}{\sin a} \int \frac{dt}{\sqrt{t}}$$

$$I = -\frac{1}{\sin a} \cdot \int t^{-1/2} \cdot dt = -\frac{1}{\sin a} \cdot t^{1/2} + C$$

$$I = -\frac{1}{\sin a} \cdot \sqrt{\cos a + \cot x \cdot \sin a} + C$$

$$I = -\frac{1}{\sin a} \sqrt{\frac{\cos a}{1} + \frac{\cos x}{\sin x} \cdot \sin a} + C$$

$$I = -\frac{1}{\sin a} \cdot \sqrt{\frac{\sin x \cdot \cos a + \cos x \sin a}{\sin x}} + C$$

Substitution,

$$\cos a + \cot x \cdot \sin a = t$$

$$0 + (-\operatorname{cosec}^2 x) \cdot dx \cdot \sin a = dt$$

$$\Rightarrow \operatorname{cosec}^2 x \cdot dx = \frac{-dt}{\sin a}$$

$$I = -\frac{1}{\sin a} \cdot \sqrt{\frac{\sin(x+a)}{\sin x}} + C$$

Q.19 $I = \int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} \cdot dx, x \in [0, 1]$

$$\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x} = \frac{\pi}{2}$$

★ $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(By Property)

$$\cos^{-1} \sqrt{x} = \frac{\pi}{2} - \sin^{-1} \sqrt{x}$$

$$I = \int \frac{\sin^{-1} \sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1} \sqrt{x} \right)}{\frac{\pi}{2}} \cdot dx$$

$$I = \int \frac{2 \sin^{-1} \sqrt{x} - \frac{\pi}{2}}{\frac{\pi}{2}} \cdot dx$$

$$I = \frac{4}{\pi} \left(\int \sin^{-1} \sqrt{x} \cdot dx - \int 1 \cdot dx \right) \rightarrow x$$

I_1

$$I_1 = \int \sin^{-1}(\sqrt{x}) \cdot dx$$

$$I_1 = \int \overset{I}{\theta} \cdot \overset{II}{\sin 2\theta} \cdot d\theta$$

ILATE

Int. by Parts

Substitution

$$\sqrt{x} = \sin \theta$$

$$\cos \theta = \sqrt{1 - \sin^2 \theta}$$

$$\sin^{-1} \sqrt{x} = \theta$$

$$\cos \theta = \sqrt{1 - x}$$

$$x = \sin^2 \theta$$

$$dx = 2 \sin \theta \cdot \cos \theta \cdot d\theta$$

$$dx = \sin 2\theta \cdot d\theta$$

$$I_1 = \theta \cdot \frac{-\cos 2\theta}{2} - \int 1 \cdot \left(\frac{-\cos 2\theta}{2} \right) \cdot d\theta$$

$$I_1 = -\frac{1}{2} \theta \cos 2\theta + \frac{1}{2} \int \cos 2\theta \cdot d\theta$$

$$I_1 = -\frac{\theta \cos 2\theta}{2} + \frac{\sin 2\theta}{4}$$

$$I_1 = -\frac{\theta \cos 2\theta}{2} + \frac{\sin 2\theta}{4}$$

$$I_1 = -\frac{\theta (1 - 2\sin^2 \theta)}{2} + \frac{2\sin \theta \cdot \cos \theta}{2}$$

$$I_1 = -\frac{\sin^{-1} \sqrt{x} \cdot (1 - 2x)}{2} + \frac{\sqrt{x} \cdot \sqrt{1-x}}{2}$$

$$I_1 = \frac{(2x-1) \cdot \sin^{-1} \sqrt{x}}{2} + \frac{\sqrt{x-x^2}}{2}$$

By eqn (1) →

$$I = \frac{4}{\pi} \int \sin^{-1} \sqrt{x} \cdot dx - \int 1 \cdot dx$$

$$\Rightarrow I = \frac{4}{\pi} \left(\frac{(2x-1) \cdot \sin^{-1} \sqrt{x}}{2} + \frac{\sqrt{x-x^2}}{2} \right) - x + C$$

$$\Rightarrow I = \frac{2}{\pi} (2x-1) \cdot \sin^{-1} \sqrt{x} + \frac{2}{\pi} \sqrt{x-x^2} - x + C$$

Q.20 $I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} \cdot dx$

$$I = -2 \int \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} \cdot \sin \theta \cdot \cos \theta \cdot d\theta$$

$$\begin{aligned} 1 - \cos \theta &= 2 \sin^2 \theta/2 \\ 1 + \cos \theta &= 2 \cos^2 \theta/2 \end{aligned}$$

$$I = -2 \int \sqrt{\frac{2 \sin^2 \theta/2}{2 \cos^2 \theta/2}} \cdot \sin \theta \cdot \cos \theta \cdot d\theta$$

$$\sqrt{x} = \cos \theta$$

$$x = \cos^2 \theta$$

$$dx = 2 \cos \theta \cdot (-\sin \theta) \cdot d\theta$$

$$dx = -2 \sin \theta \cdot \cos \theta \cdot d\theta$$

$$\begin{aligned} \sin \theta &= \sqrt{1 - \cos^2 \theta} \\ &= \sqrt{1-x} \end{aligned}$$

$$I = -2 \int \frac{\sin \theta/2}{\cancel{\cos \theta/2}} \cdot \boxed{2 \sin \frac{\theta}{2} \cdot \cancel{\cos \frac{\theta}{2}}} \cdot \cos \theta \cdot d\theta$$

$$I = -2 \int \underbrace{2 \sin^2 \theta/2}_{(1 - \cos \theta)} \cdot \cos \theta \cdot d\theta$$

$$I = (-2) \int (1 - \cos \theta) \cdot \cos \theta \cdot d\theta = \int (-2 \cos \theta + \underbrace{2 \cos^2 \theta}_{1 + \cos 2\theta}) \cdot d\theta$$

$$I = \int (-2 \cos \theta + 1 + \cos 2\theta) \cdot d\theta$$

$$I = \left(-2 \sin \theta + \theta + \frac{\sin 2\theta}{2} \right) + C$$

Resubstitute

$$I = -2 \sqrt{1-x} + \cos^{-1} \sqrt{x} + \frac{2 \sin \theta \cdot \cos \theta}{2} + C$$

$$\sqrt{x} = \cos \theta$$

$$\cos^{-1}(\sqrt{x}) = \theta$$

$$I = -2 \sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{1-x} \cdot \sqrt{x} + C$$

$$I = -2 \sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{x-x^2} + C$$

Q.21 $I = \int \left(\frac{2 + \sin 2x}{1 + \cos 2x} \right) e^x \cdot dx$

$\sin 2\theta = 2 \sin \theta \cos \theta$
 $1 + \cos 2\theta = 2 \cos^2 \theta$

$\int e^x \cdot (\underline{f(x)} + \underline{f'(x)}) \cdot dx = e^x \cdot \underline{f(x)} + C$

$I = \int \frac{2 + 2 \sin x \cdot \cos x}{2 \cos^2 x} (e^x) \cdot dx$

$I = \int e^x (\sec^2 x + \tan x) \cdot dx = e^x \cdot \tan x + C$
 (Note: $\sec^2 x$ is $f'(x)$ and $\tan x$ is $f(x)$)

Q.22 $I = \int \frac{x^2 + x + 1}{(x+1)^2 (x+2)} \cdot dx$

Partial Fraction

$\frac{x^2 + x + 1}{(x+1)^2 (x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$

$\Rightarrow x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$

$\Rightarrow x^2 + x + 1 = A(x^2 + 3x + 2) + B(x+2) + C(x^2 + 2x + 1)$

By comparison $\Rightarrow$

- $(x^2) \rightarrow 1 = A + C$ (1)
- $(x) \rightarrow 1 = 3A + B + 2C$ (2)
- $(\text{Constant}) \rightarrow 1 = 2A + 2B + C$ (3)

After solving these equations

$A = -2, B = 1, C = 3$

$$I = \int \frac{(x^2 + x + 1) \cdot dx}{(x+1)^2 (x+2)} = \int \left[\frac{-2}{(x+1)} + \frac{1}{(x+1)^2} + \frac{3}{(x+2)} \right] \cdot dx$$

$$I = -2 \log|x+1| - \frac{1}{(x+1)} + 3 \log|x+2| + C$$

Q. 23 $I = \int \tan^{-1} \sqrt{\frac{1-x}{1+x}} \cdot dx$

✓ Put $x = \cos \theta$

$\theta = \cos^{-1} x$

$\sin \theta = \sqrt{1 - \cos^2 \theta}$

$\sin \theta = \sqrt{1 - x^2}$

✓ $\Rightarrow dx = -\sin \theta \cdot d\theta$

$$I = \int \tan^{-1} \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \cdot (-\sin \theta) \cdot d\theta$$

$$I = - \int \tan^{-1} \sqrt{\frac{2 \sin^2 \theta / 2}{2 \cos^2 \theta / 2}} \cdot \sin \theta \cdot d\theta$$

$$\frac{1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}}{1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}}$$

$$I = - \int \tan^{-1} \left(\tan \frac{\theta}{2} \right) \cdot \sin \theta \cdot d\theta$$

$$I = - \frac{1}{2} \int \overset{\text{I}}{\theta} \cdot \overset{\text{II}}{\sin \theta} \cdot d\theta$$

Int. by Parts

ILATE

$$\int \text{I} \cdot \text{II} = \text{I} \cdot \int \text{II}$$

$$- \int (\text{I}' \cdot \text{II})$$

$$I = - \frac{1}{2} \left[\theta \cdot (-\cos \theta) - \int (1 - \cos \theta) \cdot d\theta \right]$$

$$I = - \frac{1}{2} \left[-\theta \cdot \cos \theta + \sin \theta \right] + C$$

$$I = \frac{1}{2} \left[+ \cos^{-1} x \cdot x - \sqrt{1 - x^2} \right] + C$$

Q.24 $I = \int \frac{\sqrt{x^2+1} \cdot [\log(x^2+1) - 2\log x]}{x^4} \cdot dx$

$$\left[\log(x^2+1) - 2\log x = \log(x^2+1) - \log x^2 = \log\left(\frac{x^2+1}{x^2}\right) \right]$$

$$= \log\left(1 + \frac{1}{x^2}\right)$$

$$I = \int \frac{\sqrt{\frac{x^2+1}{x^2}} \cdot \log\left(1 + \frac{1}{x^2}\right)}{x^3} \cdot dx$$

$$I = \int \sqrt{1 + \frac{1}{x^2}} \cdot \log\left(1 + \frac{1}{x^2}\right) \cdot \left(\frac{dx}{x^3}\right)$$

$$I = -\frac{1}{2} \int \sqrt{t} \cdot \log(t) \cdot dt$$

Int. by Parts I L A T E

$$\int I \cdot II = I \int II - \int [I' \cdot \int II]$$

$$I = -\frac{1}{2} \left[\log t \cdot \frac{2}{3} t^{3/2} - \int \frac{1}{t} \cdot \left(\frac{2}{3}\right) t^{3/2} \cdot dt \right]$$

$$I = -\frac{1}{2} \left[\left(\frac{2}{3}\right) t^{3/2} \cdot \log t - \left(\frac{2}{3}\right) \int t^{1/2} \cdot dt \right]$$

$$I = -\frac{1}{2} \cdot \frac{2}{3} \left[t^{3/2} \cdot \log t - \frac{2}{3} \cdot t^{3/2} \right] + C$$

$$I = -\frac{1}{3} \left[\left(1 + \frac{1}{x^2}\right)^{3/2} \cdot \log\left(1 + \frac{1}{x^2}\right) - \frac{2}{3} \cdot \left(1 + \frac{1}{x^2}\right)^{3/2} \right] + C$$

$$I = -\frac{1}{3} \left(1 + \frac{1}{x^2}\right)^{3/2} \cdot \left[\log\left(1 + \frac{1}{x^2}\right) - \frac{2}{3} \right] + C$$

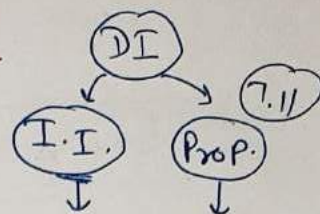
Substitution

$$1 + \frac{1}{x^2} = t$$

$$\Rightarrow \frac{-2}{x^3} \cdot dx = dt$$

$$\Rightarrow \frac{dx}{x^3} = \frac{dt}{-2}$$

Miscellaneous Exercise on Chapter (7)



Q.25

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) \cdot dx$$

$$\int e^x (f(x) + f'(x)) \cdot dx = e^x \cdot f(x) + C$$

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - 2\sin \frac{x}{2} \cos \frac{x}{2}}{2\sin^2 \frac{x}{2}} \right) \cdot dx$$

$$\begin{aligned} \checkmark 1 - \cos x &= 2\sin^2 \frac{x}{2} \\ \checkmark \sin x &= 2\sin \frac{x}{2} \cos \frac{x}{2} \end{aligned}$$

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{\operatorname{cosec}^2 \frac{x}{2}}{2} - \cot \frac{x}{2} \right) \cdot dx$$

$$I = \left[-e^x \cdot \cot \frac{x}{2} \right]_{\frac{\pi}{2}}^{\pi}$$

$$f'(x) = -(-\operatorname{cosec}^2 \frac{x}{2}) \times \frac{1}{2}$$

$$I = \left[-e^x \cot \frac{\pi}{2} \right]_0 - \left[-e^{\frac{\pi}{2}} \cot \frac{\pi}{4} \right] = +e^{\frac{\pi}{2}}$$

Q.26

$$I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} \cdot dx$$

(by dividing by $\cos^4 x$
in both Nr & Dr.)

$$I = \int_0^{\frac{\pi}{4}} \frac{(\sin x \cos x / \cos^4 x)}{(\cos^4 x + \sin^4 x) / \cos^4 x} \cdot dx$$

$$I = \int_0^{\frac{\pi}{4}} \frac{\tan x \cdot \sec^2 x}{1 + \tan^4 x} \cdot dx$$

$$I = \int_0^{\frac{\pi}{4}} \frac{\tan x \cdot \sec^2 x}{1 + (\tan^2 x)^2} \cdot dx$$

Let $\boxed{\tan^2 x = t}$ ✓

$$\Rightarrow 2(\tan x \cdot \sec^2 x \cdot dx) = dt$$

$$\Rightarrow \tan x \cdot \sec^2 x \cdot dx = \frac{dt}{2} \checkmark$$

New Limits

	Old x	New t
U.L.	$\frac{\pi}{4}$	1
L.L.	0	0

$$I = \frac{1}{2} \int_0^1 \frac{dt}{1+t^2} = \frac{1}{2} (\tan^{-1} t)_0^1$$

$$I = \frac{1}{2} [\tan^{-1} 1 - \tan^{-1} 0] = \frac{1}{2} \left[\frac{\pi}{4} - 0 \right] = \frac{\pi}{8} \checkmark$$

[Q.27] $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 \sin^2 x} \cdot dx$

by dividing $\cos^2 x$ in both No. & Dr.

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{(1 + 4 \tan^2 x)} \cdot dx$$

multiple $\sec^2 x$
in both No & Dr

$$I = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{(1 + 4 \tan^2 x) \cdot \sec^2 x} \cdot dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x \cdot dx}{(1 + 4 \tan^2 x) (1 + \tan^2 x)}$$

~~I~~ put $\boxed{\tan x = t}$

$\Rightarrow \sec^2 x \cdot dx = dt$

New

U.L. $t = \tan \frac{\pi}{2} = \infty$

L.L. $t = \tan 0 = 0$

After substitution.

$$I = \int_0^{\infty} \frac{dt}{(1+4t^2)(1+t^2)} \rightarrow \begin{matrix} \text{2 Factors} \\ \text{Partial Fraction.} \end{matrix}$$

$$\frac{1}{(1+4t^2)(1+t^2)} \Rightarrow \frac{A}{1+4t^2} + \frac{B}{1+t^2} \rightarrow \text{let } t^2 = y$$

$$\frac{1}{(1+4y)(1+y)} = \frac{A}{(1+4y)} + \frac{B}{(1+y)}$$

$$\text{Trick} \rightarrow A = \frac{4}{3} \quad B = -\frac{1}{3}$$

$$y = -\frac{1}{4} \quad y = -1$$

$$I = \int_0^{\infty} \frac{dt}{(1+4t^2)(1+t^2)} = \int_0^{\infty} \left[\frac{\frac{4}{3}}{(1+4t^2)} + \frac{-\frac{1}{3}}{(1+t^2)} \right] dt$$

$$I = \frac{4}{3} \int_0^{\infty} \frac{1}{1+(2t)^2} dt - \frac{1}{3} \int_0^{\infty} \frac{1}{1+t^2} dt$$

$$I = \frac{4}{3} \left[\frac{\tan^{-1}(2t)}{2} \right]_0^{\infty} - \frac{1}{3} \left[\tan^{-1} t \right]_0^{\infty}$$

$$I = \frac{4}{3} \left[\frac{\tan^{-1}(\infty)}{2} - \frac{\tan^{-1}(0)}{2} \right] - \frac{1}{3} \left[\tan^{-1} \infty - \tan^{-1} 0 \right]$$

$$I = \frac{4}{3} \left[\frac{\pi}{4} - 0 \right] - \frac{1}{3} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi}{6}$$

Q.28 $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} \cdot dx$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x) \cdot dx}{\sqrt{1 - \underbrace{1 + 2\sin x \cos x}_{\sin^2 x + \cos^2 x}}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x) \cdot dx}{\sqrt{1 - (1 - 2\sin x \cos x)}}$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x) \cdot dx}{\sqrt{1 - (\sin^2 x + \cos^2 x - 2\sin x \cos x)}}$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x) \cdot dx}{\sqrt{1 - (\sin x - \cos x)^2}}$$

$a^2 + b^2 - 2ab = (a-b)^2$

Let $\sin x - \cos x = t$
 $\Rightarrow (\cos x + \sin x) \cdot dx = dt$

	New
U.L.	$\frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$
L.L.	$\frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2}$

$$I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1} t \right]_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}$$

$$I = \sin^{-1} \left(\frac{\sqrt{3}-1}{2} \right) - \sin^{-1} \left(\frac{1-\sqrt{3}}{2} \right)$$

$\sin^{-1}(-x) = -\sin^{-1} x$

$$I = \sin^{-1} \left(\frac{\sqrt{3}-1}{2} \right) + \sin^{-1} \left(\frac{\sqrt{3}-1}{2} \right) = 2 \sin^{-1} \left(\frac{\sqrt{3}-1}{2} \right)$$

Q29 $I = \int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$

Rationalisation

$$I = \int_0^1 \frac{1}{\sqrt{1+x} - \sqrt{x}} \times \frac{\sqrt{1+x} + \sqrt{x}}{\sqrt{1+x} + \sqrt{x}} \cdot dx$$

$$I = \int_0^1 \frac{\sqrt{1+x} + \sqrt{x}}{1} \cdot dx$$

$$\begin{aligned} a^2 - b^2 \\ &= (\sqrt{1+x})^2 - (\sqrt{x})^2 \\ &= 1+x - x = 1 \end{aligned}$$

$$I = \int_0^1 \left[(1+x)^{1/2} + (x)^{1/2} \right] \cdot dx$$

$$I = \left[\frac{(1+x)^{3/2}}{3/2} + \frac{x^{3/2}}{3/2} \right]_0^1 = \left[\frac{2}{3/2} + \frac{1}{3/2} \right] = \left[\frac{1}{3/2} \right]$$

$$I = \frac{2 \cdot 2^{3/2}}{3} = \frac{4\sqrt{2}}{3}$$

Q.30 $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} \cdot dx$

$\sin x - \cos x$ $\xrightarrow{0-\pi+\pi}$

$$I = \frac{1}{16} \int_0^{\pi/4} \frac{(\sin x + \cos x) \cdot dx}{\frac{9}{16} + 2 \sin x \cos x - 1 + 1}$$

$$I = \frac{1}{16} \int_0^{\pi/4} \frac{(\sin x + \cos x) \cdot dx}{\frac{25}{16} - (1 - 2 \sin x \cos x)}$$

$\sin^2 x + \cos^2 x$

$$I = \frac{1}{16} \int_0^{\pi/4} \frac{(\sin x + \cos x) \cdot dx}{\frac{25}{16} - [\sin^2 x + \cos^2 x - 2 \sin x \cos x]}$$

$a^2 + b^2 - 2ab$
 $\uparrow \quad \downarrow$
 $(a-b)^2$

$$I = \frac{1}{16} \int_0^{\frac{\pi}{4}} \frac{(\sin u + \cos u)}{\left(\frac{5}{4}\right)^2 - (\sin u - \cos u)^2} \cdot du \rightarrow dt$$

Substitution:

$$\boxed{\sin u - \cos u = t}$$

$$\Rightarrow (\cos u + \sin u) \cdot du = dt$$

	(t) New
U.L.	$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$
L.L.	$0 - 1 = -1$

$$I = \frac{1}{16} \int_{-1}^0 \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$$

$$\boxed{\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C}$$

$$a = \frac{5}{4}$$

$$I = \frac{1}{16} \left[\frac{1}{2 \cdot \frac{5}{4}} \log \left| \frac{\frac{5}{4} + t}{\frac{5}{4} - t} \right| \right]_{-1}^0 = \frac{1}{40} \left[\log \left| \frac{5+4t}{5-4t} \right| \right]_{-1}^0$$

$$I = \frac{1}{40} \left[\log |1| - \log \left| \frac{1}{9} \right| \right]$$

$$\log\left(\frac{1}{m}\right) = \log(m)^{-1} = -1 \cdot \log m$$

$$\boxed{I = \frac{1}{40} (\log 9)}$$

[Q.31] $I = \int_0^{\frac{\pi}{2}} \sin 2x \cdot \tan^{-1}(\sin x) \cdot dx$

$$I = \int_0^{\frac{\pi}{2}} \underbrace{2 \sin x \cos x}_{\uparrow} \cdot \tan^{-1}(\sin x) \cdot \underbrace{dx}_{\uparrow}$$

$$I = 2 \int_0^{\frac{\pi}{2}} \underbrace{\sin x}_{\uparrow} \cdot \tan^{-1}(\sin x) \cdot \underbrace{\cos x}_{\uparrow} \cdot dx$$

Let $\sin x = t$ ✓

$\cos x \cdot dx = dt$ ✓

	New (t)
U.L.	$\sin \frac{\pi}{2} = 1$ ✓
L.L.	$\sin 0 = 0$ ✓

$$I = 2 \int_0^1 t \cdot \tan^{-1}(t) \cdot dt$$

ILATE

Int. by Parts

$$\int I \cdot II = I \int II - \int (I' \cdot \int II)$$

$$I = 2 \left[\tan^{-1} t \cdot \frac{t^2}{2} - \int \left[\frac{1}{1+t^2} \cdot \frac{t^2}{2} \right] \cdot dt \right]_0^1$$

$$I = \frac{2}{2} \left[t^2 \cdot \tan^{-1} t - \int \frac{t^2 + 1 - 1}{1+t^2} \cdot dt \right]_0^1$$

$$I = \left[t^2 \cdot \tan^{-1} t - \int \left(1 - \frac{1}{1+t^2} \right) \cdot \underline{dt} \right]_0^1$$

$$I = \left[t^2 \cdot \tan^{-1} t - t + \tan^{-1} t \right]_0^1$$

$$I = \left[\underline{\tan^{-1}(1)} - 1 + \tan^{-1}(1) \right] - [0]$$

$$I = \frac{\pi}{4} - 1 + \frac{\pi}{4} = \left(\frac{\pi}{2} - 1 \right)$$

Q.32 $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} \cdot dx$ — (1)

By King Prop. $\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$

$$I = \int_0^{\pi} \frac{(\pi-x) \cdot \tan(\pi-x)}{\sec(\pi-x) + \tan(\pi-x)} \cdot dx$$

$$\begin{aligned} \tan(\pi-x) &= -\tan x \\ \sec(\pi-x) &= -\sec x \end{aligned}$$

$$I = \int_0^{\pi} \frac{(\pi-x) \cdot \tan x}{(\sec x) + (\tan x)} \cdot dx$$
 — (2)

By eqn (1) + (2) $\rightarrow 2I = \int_0^{\pi} \frac{(x \tan x) + (\pi-x) \cdot \tan x}{(\sec x + \tan x)} \cdot dx$

$$\Rightarrow 2I = \int_0^{\pi} \frac{\pi \tan x}{(\sec x + \tan x)} \cdot \frac{(\sec x - \tan x)}{(\sec x - \tan x)} \cdot dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{(\sec x \cdot \tan x - \tan^2 x)}{(\sec^2 x - \tan^2 x)} \cdot dx$$

$$\Rightarrow I = \frac{\pi}{2} \int_0^{\pi} [\sec x \cdot \tan x - (\sec^2 x - 1)] \cdot dx$$

$$I = \frac{\pi}{2} \left[\sec x - \tan x + x \right]_0^{\pi}$$

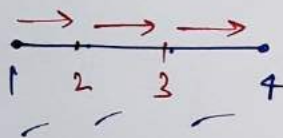
$$I = \frac{\pi}{2} \left(\left[\overset{-1}{\sec \pi} - \overset{0}{\tan \pi} + \pi \right] - \left[\underset{1}{\sec 0} - \underset{0}{\tan 0} + 0 \right] \right)$$

$$I = \frac{\pi}{2} \left[-1 + \pi - 1 \right]$$

$$I = \frac{\pi}{2} [\pi - 2] \quad \checkmark$$

[Q.33] $I = \int_1^4 (|x-1| + |x-2| + |x-3|) \cdot dx$

$x-1=0 \quad x-2=0 \quad x-3=0$
 $x=1 \quad x=2 \quad x=3$



$$I = \int_1^2 (|x-1| + |x-2| + |x-3|) \cdot dx + \int_2^3 (|x-1| + |x-2| + |x-3|) \cdot dx + \int_3^4 (|x-1| + |x-2| + |x-3|) \cdot dx$$

$$\Rightarrow I = \int_1^2 [(x-1) - (x-2) - (x-3)] \cdot dx + \int_2^3 [(x-1) + (x-2) - (x-3)] \cdot dx + \int_3^4 [(x-1) + (x-2) + (x-3)] \cdot dx$$

$$\Rightarrow I = \int_1^2 (-x + 4) \cdot dx + \int_2^3 (x) \cdot dx + \int_3^4 (3x - 6) \cdot dx$$

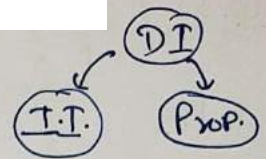
$$I = \int_1^2 (-x+4) \cdot dx + \int_2^3 (x) \cdot dx + \int_3^4 (3x-6) \cdot dx$$

$$I = \left(-\frac{x^2}{2} + 4x \right)_1^2 + \left(\frac{x^2}{2} \right)_2^3 + \left(\frac{3x^2}{2} - 6x \right)_3^4$$

$$I = \left[\underset{\uparrow}{(-2+8)} - \underset{\uparrow}{\left(-\frac{1}{2}+4\right)} \right] + \left[\underset{\uparrow}{\frac{9}{2}} - 2 \right] + \left[\cancel{4-24} - \underset{\uparrow}{\left(\frac{27}{2}-18\right)} \right]$$

$$I = 18 - \frac{17}{2} = \frac{36-17}{2} = \left(\frac{19}{2} \right) \checkmark$$

Miscellaneous Exercise on Chapter 7



Q.34 Prove that $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log \frac{2}{3}$

LHS = I = $\int_1^3 \frac{dx}{x^2(x+1)}$

Partial fraction

By Comparison,

$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$x \rightarrow 0 = A + C$

$x^2 \rightarrow 0 = A + B$

Constant

$1 = B$

$A = -1$

$C = 1$

$\Rightarrow 1 = A(x)(x+1) + B(x+1) + C(x^2)$

$\Rightarrow 1 = A(x^2+x) + B(x+1) + C \cdot x^2$

LHS = I = $\int_1^3 \frac{dx}{x^2(x+1)} = \int_1^3 \left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right) \cdot dx$

$= \left[-\log|x| - \frac{1}{x} + \log|x+1| \right]_1^3$

$= \left[-\log 3 - \frac{1}{3} + \log(4) \right] - \left[-\log(1) - 1 + \log(2) \right]$

$= (-\log 3 + \log 4 - \log 2) - \frac{1}{3} + 1$

$= \log \left(\frac{4^2}{3 \times 2} \right) + \frac{2}{3} = \log \left(\frac{2}{3} \right) + \frac{2}{3} = \underline{\underline{RHS.}}$

Q.35

$$\int_0^1 x e^x \cdot dx = 1 \quad \underline{\text{Prove.}}$$

$$\int e^x \cdot dx = e^x + c$$

$$\text{LHS} = I = \int_0^1 \overset{\text{I}}{x} \overset{\text{II}}{e^x} \cdot dx$$

ILATE

Int. by Parts.

$$\Rightarrow I = \left[x \cdot e^x - \int (1 \cdot e^x) \cdot dx \right]_0^1$$

$$\Rightarrow I = \left[x e^x - e^x \right]_0^1 = (1 \cdot e^1 - e^1) - (\underbrace{0 \cdot e^0}_{\uparrow 0} - e^0)$$

$$\Rightarrow I = +1 = \underline{\text{RHS}}$$

Q.36

$$I = \int_{-1}^1 x^{17} \cdot \cos^4 x \cdot dx = 0 \quad \leftarrow \text{Prove}$$

Prop.

$$\int_{-a}^a f(x) \cdot dx = \int_0^a [f(x) + f(-x)] \cdot dx$$

$$\text{LHS} = I = \int_{-1}^1 x^{17} \cdot \cos^4 x \cdot dx$$

$$\Rightarrow I = \int_0^1 \left(\underbrace{x^{17} \cdot \cos^4 x}_{\text{I}} + \underbrace{(-x)^{17} \cdot \cos^4(-x)}_{\text{II}} \right) \cdot dx$$

$$\Rightarrow I = \int_0^1 \left(\cancel{x^{17} \cdot \cos^4 x} - \cancel{x^{17} \cdot \cos^4 x} \right) \cdot dx = \int_0^1 (0) \cdot dx$$

$$\boxed{I = 0} = \text{RHS.}$$

Q.37 $\int_0^{\pi/2} \sin^3 x \cdot dx = \frac{2}{3}$

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

$$\Rightarrow 4 \sin^3 x = 3 \sin x - \sin 3x$$

$$\Rightarrow \sin^3 x = \frac{3 \sin x - \sin 3x}{4}$$

$$\text{LHS} = \int_0^{\pi/2} \sin^3 x \cdot dx = \int_0^{\pi/2} \left(\frac{3}{4} \sin x - \frac{1}{4} \sin 3x \right) \cdot dx$$

$$= \left[-\frac{3}{4} \cos x + \frac{1}{4} \cdot \frac{\cos 3x}{3} \right]_0^{\pi/2}$$

$$= \left[-\frac{3}{4} \cos \frac{\pi}{2} + \frac{\cos \frac{3\pi}{2}}{12} \right] - \left[-\frac{3}{4} \cos 0 + \frac{\cos 0}{12} \right]$$

$$= 0 + \frac{3}{4} - \frac{1}{12} = \frac{9-1}{12} = \frac{8}{12} = \frac{2}{3}$$

I.I

Hint.

$$\int \sin^3 x \cdot dx$$

$$= \int \sin^2 x \cdot \sin x \cdot dx$$

$$= \int (-\cos^2 x) \cdot \sin x \cdot dx$$

(+)

Q.38 $\int_0^{\pi/4} 2 \tan^3 x \cdot dx = 1 - \log 2$

$$\text{LHS} = \int_0^{\pi/4} 2 \tan^3 x \cdot dx = 2 \int_0^{\pi/4} (\tan^2 x) \cdot \tan x \cdot dx$$

$$= 2 \int_0^{\pi/4} (\sec^2 x - 1) \cdot \tan x \cdot dx$$

$$= 2 \int_0^{\pi/4} (\tan x \cdot \sec^2 x - \tan x) \cdot dx$$

$$= 2 \int_0^{\pi/4} (\underbrace{\tan x \cdot \sec^2 x}_{\text{substitution}} - \underbrace{\tan x}_{\text{substitution}}) \cdot dx$$

$$= 2 \int_0^{\pi/4} (\underbrace{\tan x}_{\substack{\downarrow \\ t}} \cdot \underbrace{\sec^2 x}_{\substack{\downarrow \\ dt}}) \cdot dx - 2 \int_0^{\pi/4} \tan x \cdot dx$$

New Limit

U.L.

$$\tan x = t$$

$$t = \tan \frac{\pi}{4} = 1$$

L.L.

$$t = \tan 0 = 0$$

$$\int \tan x \cdot dx$$

$$\downarrow$$

$$\log |\sec x| + C$$

$$= 2 \int_0^1 t \cdot dt - 2 \left[\log |\sec x| \right]_0^{\pi/4}$$

$$= 2 \left(\frac{t^2}{2} \right)_0^1 - 2 \left[\log \left| \sec \frac{\pi}{4} \right| - \log |\sec 0| \right]$$

$$= 2 \left(\frac{1}{2} - 0 \right) - 2 \left[\log |\sqrt{2}| - \log(1) \right]$$

$$= 1 - 2 \cdot \log(2)^{1/2}$$

$$= \underline{\underline{1 - \log 2}} = \text{RHS.}$$

$$\log m^n = n \cdot \log m$$

Q. 39 $\int_0^1 \sin^{-1} x \cdot dx = \frac{\pi}{2} - 1$

$\int 1 \cdot dx = \textcircled{x} + C$

LHS = $\int_0^1 \sin^{-1} x \cdot dx$ $\int \sin^{-1} x \cdot dx = I$

$I = \int \textcircled{\text{I}} \sin^{-1} x \cdot \textcircled{\text{II}} 1 \cdot dx$
I L A T E

Int. by Parts

$\int I \cdot II = \underline{I \int II} - \underline{\int (I' \cdot II)}$

$I = \cancel{\sin^{-1} x} \cdot (x) - \int \frac{1}{\sqrt{1-x^2}} \cdot (x) \cdot dx$

$I = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \cdot dx$

Subs.

$1-x^2 = t$

$\Rightarrow -2x \cdot dx = dt$

$\Rightarrow x dx = \frac{dt}{-2}$

$I = x \sin^{-1} x - \left(\frac{1}{-2}\right) \int \frac{dt}{\sqrt{t}}$

$I = x \sin^{-1} x + \frac{1}{2} \int (t^{-1/2}) \cdot dt$

$I = \left(x \sin^{-1} x + \frac{1}{2} \cdot 2 \sqrt{t} \right) + C$

$I = x \sin^{-1} x + \sqrt{1-x^2} + C = \int \sin^{-1} x \cdot dx$

LHS = $\int_0^1 \sin^{-1} x \cdot dx = \left[x \sin^{-1} x + \sqrt{1-x^2} \right]_0^1$

$= \left[\underset{\substack{\downarrow \\ \pi/2}}{\sin^{-1}(1)} + \sqrt{0} \right] - \left[0 + \sqrt{1-0} \right] = \underline{\underline{\frac{\pi}{2} - 1}} = \underline{\underline{RHS}}$

Miscellaneous Exercise on Chapter 7

[Q. 40] Evaluate $\int_0^1 e^{2-3x} \cdot dx$ as a limit of a sum.

Limit Standard Form

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1 \quad \left(\frac{0}{0} \right)$$

$$\frac{a(r^n - 1)}{r - 1}$$

$$\int_a^b \underline{f(x)} \cdot dx = \lim_{h \rightarrow 0} h \left[\underline{f(a)} + \underline{f(a+h)} + \underline{f(a+2h)} + \dots + \underline{f(a+(n-1)h)} \right]$$

$n \cdot h = b - a$

$$\int_0^1 e^{2-3x} \cdot dx \Rightarrow \text{Here } b=1, a=0 \quad n \cdot h = 1 - 0$$

$n \cdot h = 1$

$$\int_0^1 \underline{e^{2-3x}} \cdot dx = \lim_{h \rightarrow 0} h \left[\underline{f(0)} + \underline{f(h)} + \underline{f(2h)} + \dots + \underline{f((n-1)h)} \right]$$

$\uparrow$
 $f(x)$

$$= \lim_{h \rightarrow 0} h \left[e^2 + e^{2-3h} + e^{2-6h} + \dots + e^{2-3(n-1)h} \right]$$

$$= \lim_{h \rightarrow 0} h \left(e^2 \left[1 + \cancel{e^{-3h}} + \cancel{e^{-6h}} + \dots + e^{-3(n-1)h} \right] \right)$$

Sum of G.P. = $S_n = \frac{a(r^n - 1)}{r - 1}$

$a = \text{First term} = 1$

$$r = \text{C.R.} = \frac{a_2}{a_1} = \frac{e^{-3h}}{1} = e^{-3h}$$

$$\int_0^1 e^{2-3u} \cdot du = \lim_{h \rightarrow 0} h \cdot e^2 \left[1 \cdot \left\{ \frac{(e^{-3h})^n - 1}{e^{-3h} - 1} \right\} \right]$$

$$= \lim_{h \rightarrow 0} h e^2 \left[\frac{e^{-3nh} - 1}{e^{-3h} - 1} \right]$$

$$\underline{nh=1}$$

$$= e^2 \frac{(e^{-3} - 1)}{-3}$$

$$\lim_{h \rightarrow 0} \frac{-3 \cdot h}{e^{-3h} - 1}$$

Standard

Form

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

$$= \frac{e^2(e^{-3} - 1)}{-3}$$

$$= \frac{e^2(\frac{1}{e^3} - 1)}{-3} = \frac{e^2(\frac{1}{e} - e^2)}{-3}$$

$$= \frac{(e^2 - \frac{1}{e})}{3} \quad \checkmark$$

Miscellaneous Exercise on Chapter (7)

Q41 to Q44

Q.41 $\int \frac{dx}{e^x + e^{-x}}$ is equal to —

- (A) $\tan^{-1}(e^x) + c$ (B) $\tan^{-1}(e^{-x}) + c$
(C) $\log(e^x - e^{-x}) + c$ (D) $\log(e^x + e^{-x}) + c$

$$I = \int \frac{dx}{e^x + e^{-x}} = \int \frac{dx}{e^x + \frac{1}{e^x}} = \int \frac{dx}{\left(\frac{e^{2x} + 1}{e^x}\right)}$$

$$I = \int \frac{e^x}{e^{2x} + 1} \cdot dx = \int \frac{e^x \cdot dx}{(e^x)^2 + 1}$$

by substitution,

$$e^x = t \Rightarrow e^x \cdot dx = dt$$

$$I = \int \frac{dt}{t^2 + 1} = \tan^{-1}(t) + c = \underline{\underline{\tan^{-1}(e^x) + c}}$$

~~Q.42 $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ is equal to~~

Q. 42 $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ is equal to —

(A) $\frac{-1}{\sin x + \cos x} + C$ (B) $\log |\sin x + \cos x| + C$

(C) $\log |\sin x - \cos x| + C$ (D) $\frac{1}{(\sin x + \cos x)^2} + C$

$$I = \int \frac{\cos 2x}{(\sin x + \cos x)^2} \cdot dx = \int \frac{\cos^2 x - \sin^2 x}{(\sin x + \cos x)^2} \cdot dx$$

$$I = \int \frac{(\cancel{\cos x + \sin x})(\cos x - \sin x)}{(\cancel{\sin x + \cos x})^2} \cdot dx = \int \frac{\cos x - \sin x}{\sin x + \cos x} \cdot dx$$

Let $\sin x + \cos x = t$

$$\Rightarrow (\cos x - \sin x) dx = dt$$

$$I = \int \frac{dt}{t} = \log |t| + C = \log |\sin x + \cos x|$$

+ C

Q. 43 If $f(a+b-x) = f(x)$, then $\int_a^b x f(x) dx$ equals —

(A) $\frac{a+b}{2} \int_a^b f(b-x) \cdot dx$

(B) $\frac{a+b}{2} \int_a^b f(b+x) \cdot dx$

(C) $\frac{b-a}{2} \int_a^b f(x) \cdot dx$

(D) $\frac{a+b}{2} \int_a^b f(x) \cdot dx$

43) Ans. $I = \int_a^b x f(x) \cdot dx$
by King prop.

$$I = \int_a^b (\underline{a+b-x}) \cdot \underbrace{f(a+b-x)}_{f(x)} \cdot dx$$

$$I = \int_a^b (\underline{a+b-x}) \cdot \underline{f(x)} \cdot dx$$

$$I = (a+b) \int_a^b f(x) \cdot dx - \boxed{\int_a^b x \cdot f(x) \cdot dx} \rightarrow I$$

$$I = (a+b) \int_a^b f(x) \cdot dx - I$$

$$2(I) = (a+b) \int_a^b f(x) \cdot dx$$

$$\boxed{I = \frac{(a+b)}{2} \int_a^b f(x) \cdot dx}$$

King Property

$$\int_a^b \underline{f(x)} \cdot dx = \int_a^b f(\underline{a+b-x}) \cdot dx$$

Given:

$$\underline{f(a+b-x) = f(x)}$$

Q. 44 The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) \cdot dx$ is $\rightarrow$

By King Prop.

(A) 1

☒ (B) 0

(C) -1

(D) $\frac{\pi}{4}$

$$\int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx$$

$$I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) \cdot dx \quad \text{--- (1)}$$

by King Prop. $\rightarrow$ $x \rightarrow 1-x$

$$I = \int_0^1 \tan^{-1}\left(\frac{2(1-x)-1}{1+(1-x)-(1-x)^2}\right) \cdot dx$$

$$I = \int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x-x^2+2x}\right) \cdot dx$$

$$I = \int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right) \cdot dx \quad \text{--- (2)}$$

by eqn (1) + (2) $\rightarrow$

$$\tan^{-1}(-x) = -\tan^{-1}x$$

$$2I = \int_0^1 \left[\tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) + \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right) \right] \cdot dx$$

$$\Rightarrow 2I = \int_0^1 \left[\cancel{\tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)} - \cancel{\tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)} \right] \cdot dx$$

$$2I = \int_0^1 (0) \cdot dx = 0$$

$$I = 0$$